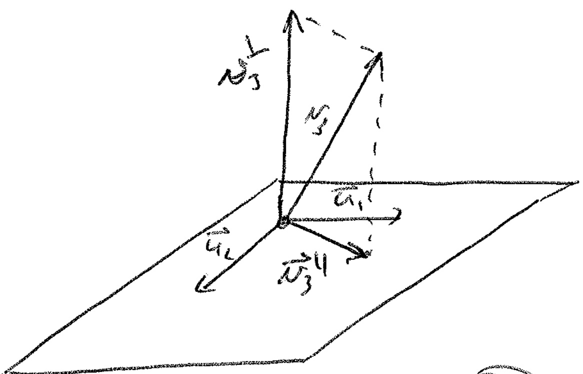
Class 22: Oct 25, 2013

(IV) cont'd

Since $\vec{v}_3 \notin W$, we can project down to it



Break up $\vec{v}_3 = \vec{v}_3^{\parallel} + v_3^{\perp}$, where

$\vec{v}_3^{\parallel} \in W = \text{span}(\vec{u}_1, \vec{u}_2)$

Then $\vec{v}_3^{\perp} = v_3 - \vec{v}_3^{\parallel} = \vec{v}_3 - \text{proj}_W \vec{v}_3$

$$= \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot v_3)\vec{u}_2$$

(V) Create $\vec{u}_3 = \dfrac{1}{\|\vec{v}_3^{\perp}\|} \vec{v}_3^{\perp}$

(VI) Continue the pattern until all remaining vectors in $\mathcal{B}$ are addressed

Result is $\mathcal{C} = \{\vec{u}_1, \dots, \vec{u}_m\}$

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ex. Problem 5.2.13

Given $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}$ as a basis for $V \subset \mathbb{R}^4$, calculate an orthonormal $\mathcal{C}$.

<u>Strategy</u> Follow steps (I)–(VI) above.

Solution Step (I), $\|\vec{v}\| = \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \sqrt{1+1+1+1} = 2$.

Hence $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$

(II) $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$$= \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \left(\begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - 1\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

Note here that $\|\vec{v}_2^{\perp}\| = 1$ already.

(III) $\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2 = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$

(IV) $\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$

$$= \begin{bmatrix}0\\2\\1\\-1\end{bmatrix} - \left(\begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}\right)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} - \left(\begin{bmatrix}1/2 & -1/2 & -1/2 & 1/2\end{bmatrix}\begin{bmatrix}0\\2\\1\\-1\end{bmatrix}\right)\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

$$= \begin{bmatrix}0\\2\\1\\-1\end{bmatrix} - (1)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} - (-2)\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$$

even $\|\vec{v}_3^{\perp}\| = 1$, so

(VI) $\vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3 = \begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$

Hence $\mathcal{C} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \right\}$

$$\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3$$

Check that ① $\|\vec{u}_1\| = \|\vec{u}_2\| = \|\vec{u}_3\| = 1$, and

② $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$.

QR Factorization

Another facet of the way one changes from a basis $\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_m\}$ of a subspace $V \subset \mathbb{R}^n$ to an orthonormal one $\mathcal{C} = \{\vec{u}_1, \cdots, \vec{u}_m\}$, is that one can keep track of the transition in a change-of-basis matrix

Q: What would the dimensions be of such a matrix?

If $\mathcal{B}$ is a basis for $V < \mathbb{R}^n$, then $\mathcal{B}$ comprises the columns of a matrix

$$M = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}, \text{ where } M[\vec{x}]_{\mathcal{B}} = \vec{x}.$$

just like in earlier chapters.

<u>Caution</u>: If $\dim(V) = m < n = \dim(\mathbb{R}^n)$, then M, of size $n \times m$, is <u>NOT</u> square, as was the case in Chapter 3.

Also, $\mathcal{C}$ is an orthonormal basis for $V < \mathbb{R}^n$, so

$$\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} = Q, \text{ where } Q[\vec{x}]_{\mathcal{C}} = \vec{x}.$$

Q: what is the way to go from $\mathcal{B}$-coords. to $\mathcal{C}$-coords.?

$$\begin{tikzcd}
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "M"] \arrow[d, "R"', "?"] & \vec{x} \\
{[\vec{x}]_{\mathcal{C}}} \arrow[ur, "Q"'] &
\end{tikzcd}$$

Is there a linear transformation R, where

$$[\vec{x}]_{\mathcal{B}} \xrightarrow{R} [\vec{x}]_{\mathcal{C}}$$

Note: We cannot do something like take $R = Q^{-1}M$ (follow M first then follow Q backwards)

Since Q is not square, and has no inverse

Note: In Chapter 3 we did something similar, but there V was not a subspace of $\mathbb{R}^n$ but <u>all</u> of $\mathbb{R}^n$. Then Q, M are square (they were called S back then) and we could find the inverse of Q. <u>Not here though</u>.

Here we can do it if we can find R so that

$$\begin{tikzcd}
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "M"] \arrow[d, "R"'] & \vec{x} \\
{[\vec{x}]_{\mathcal{C}}} \arrow[ur, "Q"']
\end{tikzcd}$$

other way

one way

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{M_{n\times m}} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_{Q_{n\times m}} \cdot \underbrace{R}_{R_{m\times m}}$$

Here, R is square and takes $\mathcal{B}$-coords to $\mathcal{C}$-coords.

<u>Thm</u> For a matrix $M_{n\times m}$ w/ linearly indep. columns $\vec{v}_1, \ldots, \vec{v}_m$, $\exists$ a matrix $Q_{n\times m}$ whose columns are orthonormal (orthogonal, unit length and linearly indep.), and a matrix $R_{m\times m}$ which is upper triangular with positive diagonal entries, such that

$$M = QR$$

Called the QR Factorization of M.

$\boxed{\text{This rep is unique}}$

Furthermore, for $R = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mm} \end{bmatrix}$

$$r_{11} = \|\vec{v}_1\|, \quad r_{jj} = \|\vec{v}_j^{\perp}\|, \quad j = 2, \dots, m,$$

$$r_{ij} = \begin{cases} \vec{u}_i \cdot \vec{v}_j, & \text{for } i < j \\ 0 & \text{for } i > j. \end{cases}$$

Note: Again, this is algorythmic and both Q and R can be constructed simultaneously.

Back to the previous example

We use the previous example and above to compute R.

(I) $\|\vec{v}_1\| = \text{length of } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 = r_{11}$

(II) $\|\vec{v}_2^{\perp}\| = \text{length of } \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = 1 = r_{22}$

(III) $\|\vec{v}_3^{\perp}\| = \text{length of } \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = 1 = r_{33}$

IV $r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1$

$$r_{13} = \vec{u}_1 \cdot \vec{v}_3 = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = 2\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right) - 1\left(\tfrac{1}{2}\right) = 1$$

$$r_{23} = \vec{u}_2 \cdot \vec{v}_3 = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = -2.$$

$$r_{21} = r_{31} = r_{32} = 0$$

Hence $R_{3\times3} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. Does it work?

Check 1: Matrix Math.

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{M} \overset{\checkmark}{=} \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$M = Q \cdot R$$

Check 2: Check a one vector: let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.
for example.

Then $M[\vec{x}]_{\mathcal{B}} = 1\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ -2 \end{bmatrix}$

Hence in the standard basis in $\mathbb{R}^4$, $\vec{x} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ -2 \end{bmatrix}$.

$$QR[\vec{x}]_{\mathcal{B}} = QR\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = Q\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{[\vec{x}]_{\mathcal{C}}} = Q\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$$

$$Q[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ -2 \end{bmatrix} = \vec{x}$$

Both paths in

$$\begin{tikzcd} {[\vec{x}]_{\mathcal{B}}} \arrow[r, "M"] \arrow[d, "R"'] & \vec{x} \\ {[\vec{x}]_{\mathcal{C}}} \arrow[ur, "Q"'] & \end{tikzcd}$$

lead to the same place on the vector $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.