Class 21 : Oct 23, 2013

Some other facts from Section 5.1

(I) Orthogonal complement - Given any linear subspace $V \subset \mathbb{R}^n$, its orthogonal complement is the set of all vectors that are orthogonal to everything in V:

$$V^{\perp} = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \quad \forall \vec{v} \in V \}.$$

Notes: (1) Any linear combination of vectors that satisfy $\vec{x} \cdot \vec{v} = 0$ also satisfy this: $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

(2) $V \cap V^{\perp} = \{\vec{0}\}$.

(3) $\dim(V^{\perp}) = n - m$, for $m = \dim(V)$. (why?)

Notes: cont'd.

(4) $V^{\perp}$ is precisely the kernel of $T: \mathbb{R}^n \to \mathbb{R}^n$, the orthogonal projection of $\mathbb{R}^n$ onto a subspace $V \subset \mathbb{R}^n$, given by $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^{\parallel}$. Here

$$\text{im}(T) = V, \quad \ker(T) = V^{\perp}$$

(5) ex. Suppose $V \subset \mathbb{R}^3$, $V = \{(x,y,0) \in \mathbb{R}^3\}$, the "floor" of 3-space. Any vector of the form $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ is orthogonal to everything in V (for any given element of V, there may be others. But as a set, $V^{\perp}$ consists of vectors orthogonal to everything in V.

$$V^{\perp} = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \vec{x} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}, \; c \in \mathbb{R} \right\}$$

a line (the vertical line in 3-space).

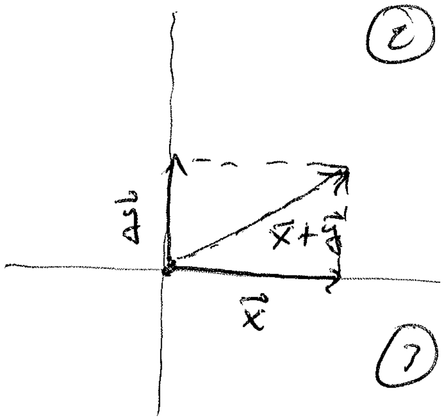
(II) Pythagorean Thm

Given 2 vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, the equation

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

holds iff $\vec{x} \cdot \vec{y} = 0$ ($\vec{x}$ and $\vec{y}$ are orthogonal)

Notes ① Any 2 orthogonal vectors in $\mathbb{R}^n$ span a plane (a 2-dim subspace of $\mathbb{R}^n$)



② In this plane, the 2 vectors form the sides of a rectangle. The diagonal of this rectangle is the vector $\vec{x}+\vec{y}$.

③ The opposite diagonal is the hypotenuse of the right triangle w/ sides $\vec{x}$ and $\vec{y}$.

④ Both diagonals have the same length. By Pythagorean

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

III Projections lose magnitude

Given any projection $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^{\parallel}$ onto $V \subset \mathbb{R}^n$, we have

$$\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$$

always, and only equality when $\vec{x} \in V$.

The reason is precisely Pythagorean's Thm again, and the fact that the hypotenus of a right triangle is ALWAYS greater than each of the sides.

IV Cauchy Schwarz Inequality

For $\vec{x}, \vec{y} \in \mathbb{R}^n$, $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|y\|$

with equality iff $\vec{x}, \vec{y}$ are parallel.

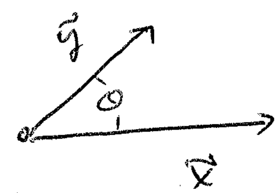
Note: if 2 vectors are parallel (remember that we always represent them based at the origin of $\mathbb{R}^n$) then they are multiples of each other (span the same line).

(V) Angle between 2 vectors

Any 2 non-colinear vectors in $\mathbb{R}^n$ span a plane, and form an angle with respect to each other in this plane.

Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\vec{x} \neq k\vec{y}$, $\forall k \in \mathbb{R}$, then

$$\theta = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$$



is the angle between them.

Why is this true? In general, the "geometric" form of the dot product of 2 vectors in the plane is

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

where θ is the angle between them.

Compare formulas when $\theta = 0, \frac{\pi}{2}, \pi$ and such.

Another question

Given a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ of an m-dim. subspace V of $\mathbb{R}^n$, how can one construct an orthonormal basis?

There is a procedure called the Gram-Schmidt process for turning $\mathcal{B}$ into a new orthonormal basis. $\mathcal{O}$

Note: Book uses the Fraktur capital letter A, or $\mathfrak{A}$. I will always use $\mathcal{O}$ for the Orthonormal basis.

Notes ① This procedure is done vector by vector across the basis, and involves a 2 stage process:

ⓐ Creation of a vector orthogonal to all previous ones, and

ⓑ Normalize it so that its norm is 1.

Process

Given $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ of a m-dimensional subspace $V < \mathbb{R}^n$, we create an orthonormal basis $\mathcal{C} = \{\vec{u}_1, \dots, \vec{u}_m\}$ as follows:

(I) Create $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$ by normalizing $\vec{v}_1$ (dividing $\vec{v}_1$ by its length to create a unit vector $\vec{u}_1$).

(II) Notice that $\text{span}(\vec{u}_1) = L$, a line in $\mathbb{R}^n$. Notice also that $\vec{v}_2 \notin L$ (it cannot be; why?)

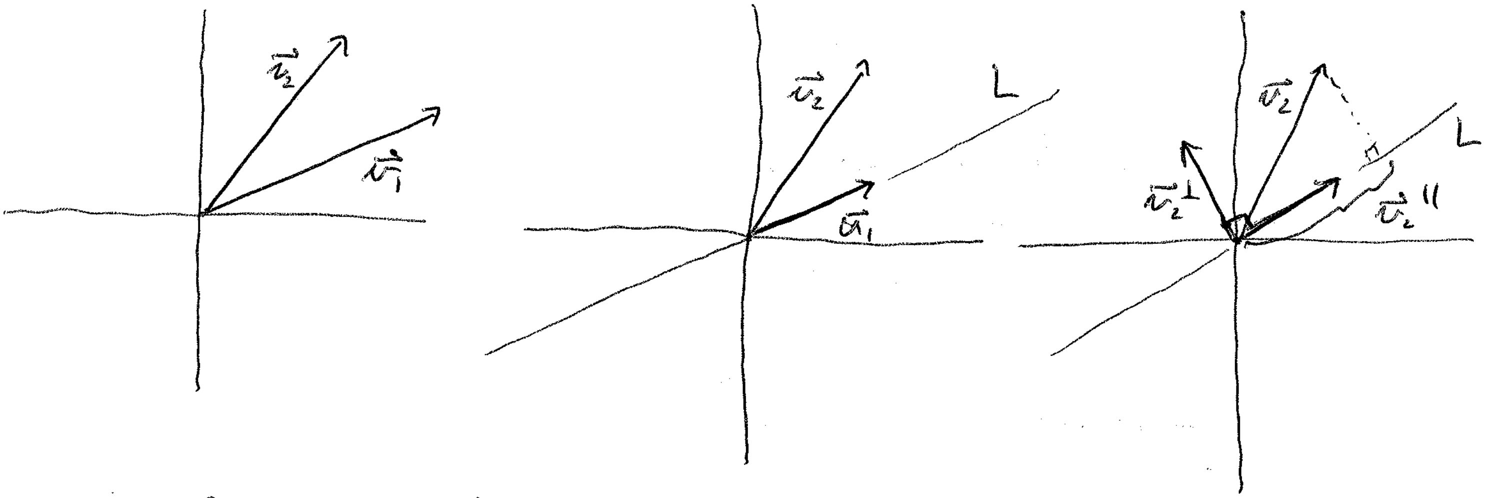
Break up $\vec{v}_2 = \vec{v}_2^{\,\parallel} + \vec{v}_2^{\,\perp}$, where $\vec{v}_2^{\,\parallel} \in \underbrace{\text{span}(\vec{u}_1)}_{=L}$ and $\vec{v}_2^{\,\perp}$ is perpendicular to L.

Then
$$\vec{v}_2^{\,\perp} = \vec{v}_2 - \vec{v}_2^{\,\parallel} = \vec{v}_2 - \text{proj}_L \vec{v}_2$$
$$= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

by the definition of a orth. projection in Section 5.1.

(II) cont'd.

Notice that $\vec{v}_1^{\perp}$ is orthogonal to $\vec{u}_1$.



Calculate $\vec{v}_2$ in this way.

(III) Create $\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}$ by normalizing $\vec{v}_2^{\perp}$.

Here, $\vec{u}_1, \vec{u}_2$ are part of $\mathcal{C}$, and are orthonormal.

They span a plane $W \subset \mathbb{R}^n$. This plane __is__ the ~~(IV) Notice that~~ same plane spanned by $\vec{v}_1, \vec{v}_2$ but $\vec{u}_1, \vec{u}_2$ are orthonormal.

(IV) Notice that $\vec{v}_3 \notin W$ (it cannot be: why?)