Class 20: Oct 21, 2013

Chapter 5: Back to $\mathbb{R}^n$ and its "points".

Recall (I) 2 vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ are called <u>orthogonal</u> or <u>perpendicular</u> if $\vec{v}_1 \cdot \vec{v}_2 = 0$.

(II) The length of a vector $\vec{u} \in \mathbb{R}^n$ is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

(III) A vector $\vec{u}$ is called a unit vector if $\|\vec{u}\| = 1$

And any vector $\vec{v} \in \mathbb{R}^n$ is a multiple of a unit vector $\vec{u}$, where $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

(Note: Any <u>nonzero</u> vector $\vec{v} \in \mathbb{R}^n$). (why?)

ex. let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Here $\vec{v}_1$ and $\vec{v}_2$ are perpendicular since $\vec{v}_1 \cdot \vec{v}_2 = [1\ 0\ 1\ 0]\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$.

And $\|\vec{v}_2\| = 1$, while

$$\|\vec{v}_1\| = \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \sqrt{[1\ 0\ 1\ 0]\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} = \sqrt{1+1} = \sqrt{2}.$$

Def A set of vectors $\vec{u}_1, \cdots, \vec{u}_m \in \mathbb{R}^n$ are called <u>orthonormal</u> if for $i = 1, \cdots, m$, $\|\vec{u}_i\| = 1$, and

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Notes Orthonormal vectors are

① <u>linearly independent</u> – Any relation involving one as a linear combination of others would necessarily be trivial.

② If there are n of them in $\mathbb{R}^n$, they form a basis

ex1: $\vec{e}_1, \cdots, \vec{e}_n \in \mathbb{R}^n$ form an orthonormal basis.

ex2: $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^3$.

ex3: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^4$ but not an orthonormal one.

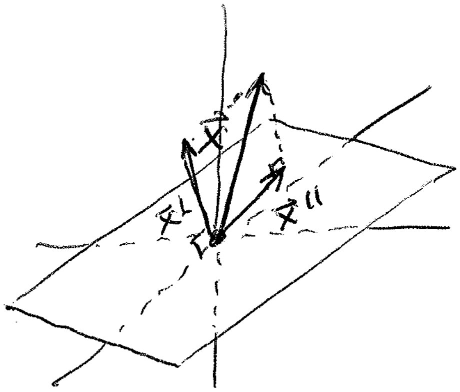
The geometry of vectors (their size, direction and other "visual" aspects of them) allows us to say important things about the spaces they "live in" and the linear transformations on those spaces.

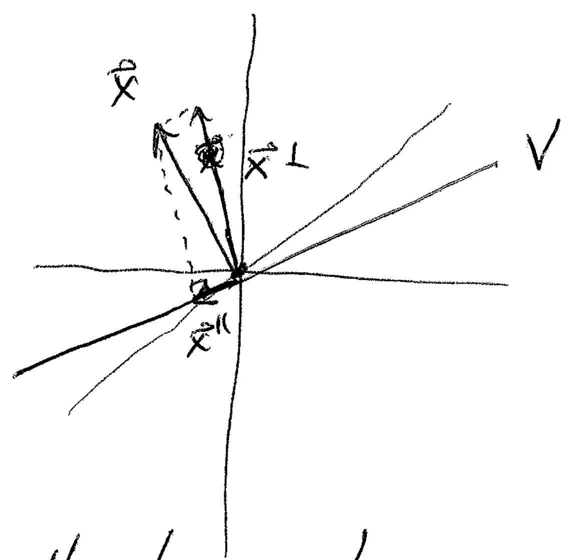
One example is for orthogonal projections

Def. Given a vector $\vec{x} \in \mathbb{R}^n$ and any m-dim subspace $V \subset \mathbb{R}^n$, we can $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel} \in V$, and $\vec{x}^{\perp}$ is perpendicular to everything in V. This way to write $\vec{x}$ as a sum of vectors, one in V and one perpendicular to V is unique.

Notes: ① The subspace V, being m-dimensional in $\mathbb{R}^n$ means it can be any dimension from 0-dim (the origin), to $(n-1)$-dimensional (a hyperplane) and anything in between. However, being a subspace, $\vec{0} \in V \subset \mathbb{R}^n$ always.

② ex. V is a plane in $\mathbb{R}^3$ $\qquad$ V is a line in $\mathbb{R}^3$





Here, on the left, $\vec{x}^\perp$ will always be a vector $\perp V$. This is a 1-dim space.

On the right, $\vec{x}^\perp$ will live in a 2-dim space of all vectors $\perp V$. Can you see this?

<u>Notes</u>: cont'd.

(3) We call $\vec{x}^{\,\parallel}$ the orthogonal projection of $\vec{x}$ onto V, and denote it $\text{proj}_V \vec{x}$.

(4) The transformation $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^{\,\parallel}$ is linear on $\mathbb{R}^n$.

Q: How can we compute a projection in $\mathbb{R}^n$ to some subspace V?

A: Via a good orthonormal basis for $V < \mathbb{R}^n$.

Think of it this way: Given $\vec{u}_1, \dots, \vec{u}_m \in V$ an $\underbrace{\text{orthonormal}}_{\text{perp. of norm 1}}$ basis, the lin. transf.

$$T : \mathbb{R}^n \to V, \quad T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}^{\,\parallel}$$

makes

$$T(\vec{x}) = \vec{x}^{\,\parallel} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_m \vec{u}_m$$

since $\vec{x}^{\,\parallel} \in V$ and $\vec{u}_1, \dots, \vec{u}_m$ is a basis for V.

Note: The constants $c_1, \dots, c_m \in \mathbb{R}$ are the coordinates of $\vec{x}^{\parallel}$ in the orthonormal basis on V.

We also know that $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$ is orthogonal to everything in V (dot product with anything in V is 0).

Hence for any $i = 1, \dots, m$, $\vec{x}^{\perp} \cdot \vec{u}_i = 0$.

Then
$$\vec{x}^{\perp} \cdot \vec{u}_i = 0 = (\vec{x} - \vec{x}^{\parallel}) \cdot \vec{u}_i$$
$$= (\vec{x} - (c_1\vec{u}_1 + \dots + c_m\vec{u}_m)) \cdot \vec{u}_i$$
$$= \vec{x} \cdot \vec{u}_i \underbrace{- c_1(\vec{u}_1 \cdot \vec{u}_i) - \dots - c_m(\vec{u}_m \cdot \vec{u}_i)}_{\substack{\text{all } 0 \text{ except for} \\ -c_i(\vec{u}_i \cdot \vec{u}_i) = -c_i \cdot 1 \\ = -c_i}}$$

$$0 = (\vec{x} \cdot u_i) - c_i$$

$$\Longrightarrow \quad c_i = \vec{x} \cdot \vec{u}_i$$

We can use this to rewrite $\vec{x}^{\parallel} = c_1\vec{u}_1 + \dots + c_m\vec{u}_m$

as $$\vec{x}^{\parallel} = \underbrace{(\vec{x} \cdot \vec{u}_1)}_{\text{scalar}} \vec{u}_1 + \dots + \underbrace{(\vec{x} \cdot \vec{u}_m)}_{\text{scalar.}} \vec{u}_m$$

Thm The linear transformation $T: \mathbb{R}^n \to V$, for $V \subset \mathbb{R}^n$ an ~~subspace~~ m-dimensional subspace, given by

$$T(\vec{x}) = \operatorname{proj}_V \vec{x} = \vec{x}^{\parallel}$$

is given by

$$T(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$
$$= \sum_{i=1}^{m} (\vec{u}_i \cdot \vec{x})\vec{u}_i$$

where $u_1, \dots, u_m$ is an orthonormal basis for V.

---

ex. Let $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}$. Find the projection of $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ onto V.

<u>Strategy</u>: Find an orthonormal basis for V and then use the previous theorem.

Solution: We already know V is spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. They are also linearly independent since $\vec{v}_1 \cdot \vec{v}_2 = [1\ 0\ 1\ 0]\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$.

Hence $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ form a basis for V in $\mathbb{R}^4$. (See example pg 1 here).

However, $\mathcal{B}$ is NOT an orthonormal basis (why not?). We form an orthonormal basis

$$\vec{u}_1 = \frac{\vec{v}_1}{\|v_1\|}, \quad \vec{u}_2 = \frac{\vec{v}_2}{\|v_2\|}$$

$$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \{\vec{u}_1, \vec{u}_2\}.$$

Now for $T: \mathbb{R}^4 \to V$, $T(\vec{x}) = \vec{x}^{\parallel}$,

$$T\begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2$$

$$= \left(\left[\tfrac{1}{\sqrt{2}}\ 0\ \tfrac{1}{\sqrt{2}}\ 0\right] \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} + \left([0\ 1\ 0\ 0] \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution cont'd.

$$T\begin{bmatrix}2\\3\\2\\1\end{bmatrix} = \left(\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}}\right)\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\2\\0\end{bmatrix} + \begin{bmatrix}0\\3\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\\2\\0\end{bmatrix}$$

Hence for $\vec{x} = \begin{bmatrix}2\\3\\2\\1\end{bmatrix}$, $\vec{x}^{\parallel} = \text{proj}_V \vec{x} = \begin{bmatrix}2\\3\\2\\0\end{bmatrix}$.

Note: In coordinates, we get

$\mathcal{B}$-basis: $\left[T\begin{bmatrix}2\\3\\2\\1\end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$

$\mathcal{C}$-basis: $\left[T\begin{bmatrix}2\\3\\2\\1\end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}4/\sqrt{2}\\3\end{bmatrix}$.

Only the second ($\mathcal{C}$) is orthonormal.