Class 19: Oct 18, 2013

Last 2 examples from Chapter 4

ex. Define a new linear map

$$R : P_2 \to P_2, \quad R(f) = f + 2f' - f''$$

Show this map is an isomorphism.

<u>Strategy</u>: Use the coordinate isomorphism to produce the equivalent linear transformation of $\mathbb{R}^3$. The matrix for that lin. transf. will answer the question.

<u>Solution</u> We can look for the matrix B so that $R[f]_{\mathcal{B}} = B[f]_{\mathcal{B}}$ relative to the basis $\mathcal{B} = \{1, x, x^2\}$ for P_2.

We can calculate this directly:

Recall that $[f]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ for $\mathcal{B} = \{1, x, x^2\}$

and $f(x) = a + bx + cx^2$.

$$B[f]_{\mathfrak{B}} = [R(f)]_{\mathfrak{B}} = [f + 2f' - f'']_{\mathfrak{B}}$$

since coordinates behave linearly (Thm. 4.1.4)

$$= [f]_{\mathfrak{B}} + 2[f']_{\mathfrak{B}} - [f'']_{\mathfrak{B}}$$

b/c $f(x) = a + bx + cx^2$, $f'(x) = b + 2cx$, $f''(x) = 2c$

$$= \begin{bmatrix} a \\ b \\ c \end{bmatrix} + 2\begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix} - \begin{bmatrix} 2c \\ 0 \\ 0 \end{bmatrix}.$$

$$= \begin{bmatrix} a + 2b - 2c \\ b + 4c \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{[f]_{\mathfrak{B}}}.$$

Hence $B = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, and since $\operatorname{rank}(B) = 3$ (you should show this), R is an isomorphism.

<u>Thm</u> Given a linear transformation $T: V \to V$ of a linear space V w/ basis $\mathfrak{B} = \{f_1, \dots, f_n\}$, the matrix B relative to $\mathfrak{B}$ is given by

$$B = \begin{bmatrix} | & & | \\ [T(f_1)]_{\mathfrak{B}} & \cdots & [T(f_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}$$

Note: ① This is precisely the same formulation as in Thm 3.4.3 of Chapter 3. The only difference here is V may not be a subspace of $\mathbb{R}^n$ (it may be an abstract linear space).

② When $V = \mathbb{R}^n$, and the basis is the standard basis, then this is precisely the same formulation as Thm 2.1.2 of Chapter 2, with $T: \mathbb{R}^n \to \mathbb{R}^n$ and basis $\{\vec{e}_1, \dots, \vec{e}_n\}$.

Back to previous example, with $R: P_2 \to P_2$,
$R(f) = f + 2f' - f''$.
Let's reconstruct the matrix B relative to $\mathfrak{B} = \{1, x, x^2\}$ but use the theorem; also

$$B = \begin{bmatrix} | & | & | \\ [R(1)]_{\mathfrak{B}} & [R(x)]_{\mathfrak{B}} & [R(x^2)]_{\mathfrak{B}} \\ | & | & | \end{bmatrix}$$

Here

- $R(1) = (1) + 2(1)' - (1)'' = 1 + 0x + 0x^2$

  So $[R(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- $R(x) = (x) + 2(x)' - (x)'' = x + 2 = 2 + 1x + 0x^2$

  So $[R(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

- $R(x^2) = (x^2) + 2(x^2)' - (x^2)'' = x^2 + 4x - 2$

  So $[R(x^2)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$

$\Longrightarrow$ $B = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ as before

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One final problem

Consider the same map but on all functions

$$R : C^{\infty}(\mathbb{R}) \longrightarrow C^{\infty}(\mathbb{R}), \quad R(f) = f + 2f' - f''$$

Let V be the subspace of $C^{\infty}(\mathbb{R})$ consisting of all functions of the form $f(x) = a\sin x + b\cos x$ (linear combinations of sines and cosines).

Find the matrix B relative to $\mathcal{B} = \{\sin x, \cos x\}$ of V.

Notes: ① for any function of the form $f(x) = a\sin x + b\cos x$, $R(f)$ will also be a function of this form (check this).

Therefore, R can be seen as a lin trans. $R: V \to V$. We look for the matrix B of this map (it is finite dimensional, while $R: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is not!)

② This space $V = \text{span}\{\sin x, \cos x\} \subset C^\infty(\mathbb{R})$ is very important in applications. Any function which is a pure sine wave is of this form, with the values of a and b determined by "when" one starts the sine wave:

Let $g(x) = \sin(x+c)$, a "shifted" wave.

Then $g(x) = \sin x \underbrace{\cos c}_{a} + \cos x \underbrace{\sin c}_{b}$ by trig.

$= a\sin x + b\cos x$

Solution: The coordinate map here is

$$L_{\mathcal{B}} : V \to \mathbb{R}^2, \quad L_{\mathcal{B}}(f) = L_{\mathcal{B}}(a\sin x + b\cos x) = \begin{bmatrix} a \\ b \end{bmatrix}$$

We compute B via the theorem:

- $$B = \begin{bmatrix} | & | \\ [R(\sin x)]_{\mathcal{B}} & [R(\cos x)]_{\mathcal{B}} \\ | & | \end{bmatrix}, \text{ for}$$

$\mathcal{B} = \{\sin x, \cos x\}$ the basis of V.

Here

- $$\begin{aligned} R(\sin x) &= (\sin x) + 2(\sin x)' - (\sin x)'' \\ &= \sin x + 2\cos x + \sin x \\ &= 2\sin x + 2\cos x \end{aligned}$$

So $[R(\sin x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

- $$\begin{aligned} R(\cos x) &= (\cos x) + 2(\cos x)' - (\cos x)'' \\ &= \cos x - 2\sin x + \cos x \\ &= -2\sin x + 2\cos x \end{aligned}$$

So $[R(\cos x)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

And thus $B = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$.

Why is this important?

Suppose in the previous problem, we wanted to know what function $f(x)$ has ~~f~~ as its image $R(f)$ the function ~~f~~ $\sin x - \cos x$.

This is the question:

~~Solve~~ Find f, where

$$(*) \qquad f + 2f' - f'' = \sin x - \cos x$$

This is a 2nd order, linear, constant-coefficient non-homogeneous differential equation.

Somewhere in the middle of 110.302 Differential Equations, we learn how to solve this (find a $f(x)$ that satisfies the equation) via a technique called

Undetermined Coefficients

along with some integration. It involves a fair amount of calculus.

However, in linear algebra, with a linear space like $C^{\omega}(\mathbb{R})$ and a subspace like $V = \text{span}\{\sin x, \cos x\}$, and the transformation $R: V \to V$, $R(f) = f + 2f' - f''$, we can interpret (*) above as simply a lin algebra problem:

Find $f(x) = a\sin x + b\cos x$, $[f]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathcal{B} = \{\sin x, \cos x\}$

so that $R[f]_{\mathcal{B}} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} [f]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

since $[\sin x - \cos x]_{\mathcal{B}} = [1\sin x - 1\cos x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus we solve $\underbrace{\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}}_{B} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\begin{bmatrix} a \\ b \end{bmatrix}$.

Here $B^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$, so $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$.

Hence $f(x) = 0\sin x - \frac{1}{2}\cos x = -\frac{1}{2}\cos x$ solves (*).

Check this: For $P(x) = -\frac{1}{2}\cos x$

$$R(P) = \left(-\tfrac{1}{2}\cos x\right) + 2\left(-\tfrac{1}{2}\cos x\right)' - \left(-\tfrac{1}{2}\cos x\right)''$$
$$= -\tfrac{1}{2}\cos x + \sin x - \tfrac{1}{2}\cos x$$
$$= \sin x - \cos x \qquad \checkmark.$$

Linear Algebra, in this abstract way, is a vital tool in understanding lots of situations, spaces, and concepts that do not look at first glance to be relevant.

Understanding this power is a very smart idea for scientists and engineers.