Class 18: 10/16/13

Recall from last class, any invertible linear transformation is an isomorphism. And if $T: V \to W$ is an isomorphism, then we call V and W isomorphic (and sometimes write $V \cong W$).

Some Consequences of Isomorphisms

(I) If $T: V \to W$ is an isomorphism, then
  (a) $\ker(T) = \{0\}$, and $\mathrm{im}(T) = W$
  (2) $\dim(V) = \dim(W)$

(II) Suppose for any $T: V \to W$, we knew $\ker(T) = \{0\}$. If $\dim(V) = \dim(W)$, then T is an isomorphism

(III) Suppose for any $T: V \to W$, we knew $\mathrm{im}(T) = W$. If $\dim(V) = \dim(W)$, then T is an isomorphism.

ex. Show $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$ is an isomorphism.

Solution: $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is invertible, so it is true. But to reinforce, we know $\dim(\mathbb{R}^2) = \dim(\mathbb{R}^2)$. And since $\ker(T) = \ker(A) = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$, we are done by (II).

ker(A) = set of solutions to $\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

The equations are $\left.\begin{matrix}2x+y=0\\x+y=0\end{matrix}\right\}$ solved only by $\begin{matrix}x=0\\y=0\end{matrix}$

(show this!).

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ex. Define $T_3: P_3 \longrightarrow \mathbb{R}^2$ by $T(f) = \begin{bmatrix}f(1)\\f(2)\\f(3)\end{bmatrix}$

Q: Is T_3 an isomorphism?

A: Absolutely not, as $\dim(P_3) = 4 \neq 3 = \dim(\mathbb{R}^3)$

This is the contrapositive of Statement (I) above:

If $\dim(V) \neq \dim(W)$, then $T: V \to W$ is not an isomorphism.

Q: How about $T_2: P_2 \longrightarrow \mathbb{R}^3$, $T(f) = \begin{bmatrix}f(1)\\f(2)\\f(3)\end{bmatrix}$

Note: This is nothing but a simple example, and asks: Is every quadratic polynomial unique determined by its evaluation on $x=1$, $x=2$, and $x=3$?

The short, clever answer to this question is:

Yes! why?

(A) Here $\dim(P_2) = \dim(\mathbb{R}^3)$ so at least there is hope that T_2 is an isomorphism.

(B) $\ker(T_2)$ is set of all solutions to

$$f(1) = 0, \quad f(2) = 0, \quad f(3) = 0$$

But every $f \in P_2$ can only have at most 2 zeros (it is a quadratic). Hence the ONLY function that can have zeros at $x=1$, $x=2$, and $x=3$ is $f(x) \equiv 0$.

Hence $\ker(T_2) = \{0\}$,

By (II) above, T_2 is an isomorphism.

The long answer involves more work, and we will need another couple of maps.

Consider the map $T_1 : P_2 \to P_2$, $T_1(f) = f(1) + f(2)x + f(3)x^2$.

Hard to tell if this is an isomorphism or not since both domain and codomain are abstract linear spaces.

Hence it is hard to think of pts in P_2 as vectors and work with matrices.

One can show that T_1 is linear (do this!) and then try to construct $\ker(T_1)$ (do this also!)

But there is a better way!

<u>Def.</u> Call the linear map $L_{\mathcal{B}} : P_2 \to \mathbb{R}^3$ defined by $L_{\mathcal{B}}(f) = \begin{bmatrix} f(0) \\ f'(0) \\ \frac{1}{2}f''(0) \end{bmatrix}$ the coordinate map of P_2 relative to the basis $\mathcal{B} = \{1, x, x^2\}$. $L_{\mathcal{B}}$ is an isomorphism, so that $P_2 \cong \mathbb{R}^3$.

Why is $L_{\mathcal{B}}$ called the coordinate map relative to $\mathcal{B}$, or the $\mathcal{B}$-coordinate map?

Because under L, we have

$$f(x) = a + bx + cx^2 \xrightarrow{L_{\mathcal{B}}} \begin{bmatrix} f(0) = a \\ f'(0) = b \\ \frac{1}{2}f''(0) = c \end{bmatrix}$$

This is the map that takes any $f \in P_2$ to the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ whose entries are the coordinates of f relative to $\mathcal{B} = \{1, x, x^2\}$:

$$L(f) = [f]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Note: $L_{\mathcal{B}}$ is an isomorphism since

ⓐ $\dim P_2 = \dim(\mathbb{R}^3) = 3$, and

ⓑ $\ker(L_{\mathcal{B}})$ = set of all f where

$$\left.\begin{array}{l} i)\ f(0) = 0 = a \\ ii)\ f'(0) = 0 = b \\ iii)\ \frac{1}{2}f''(0) = 0 = c \end{array}\right\} \begin{array}{l} f(x) \equiv 0 \\ \underline{\text{only soln.}} \end{array}$$

<u>Remark</u>: Now we can simply consider P_2 to <u>be</u> the vector space $\mathbb{R}^3$ through the isomorphism $L_{\mathcal{B}}$.

Now let's go back to our original question:

Is $T_1 : P_2 \to P_2$, $T_1(f) = f(1) + f(2)x + f(3)x^2$ an isomorphism?

To answer this, we use the coordinate map:

$$\begin{tikzcd}
f(x) \arrow[rr, "T_1"] \arrow[d, "L_{\mathcal{B}}"] & & f(1)+f(2)x+f(3)x^2 \arrow[d, leftrightarrow, "L_{\mathcal{B}}."] \\
{[f(x)]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \arrow[rr, "T_4"] & & \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix}
\end{tikzcd}$$

$$\begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+2b+4c \\ a+3b+9c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = B\,[f]_{\mathcal{B}}$$

In essence we use the coordinate map $L_{\mathcal{B}}$ to redraw the map T_1 in terms of vectors in $\mathbb{R}^3$ as a new map $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$T_4(\vec{x}) = B\vec{x}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

Here, B is invertible (show that $\mathrm{rref}(B) = I_3$).

Hence T_4 is an isomorphism.

But then so is $L_{\mathcal{B}}$, and hence $L_{\mathcal{B}}^{-1}$.

In fact, (A) $T_2 = L_{\mathcal{B}}^{-1} \circ T_4 \circ L_{\mathcal{B}}$ and as all three on the right hand side are isomorphisms, so is T_2.

(B) In the basis, $\mathcal{B}$, we can view T_2 as the matrix $\mathcal{B}$ in T_4.