Class 17: 10/15/13

One last point from Section 4.1

Def A linear space V is called finite-dimensional if it has a finite basis $f_1, \dots, f_n$ so that $\dim(V) = n$. Otherwise, V is called infinite dimensional.

ex. Denote by P the space of all polynomials in $F(\mathbb{R}, \mathbb{R})$ (the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$).

① P is a linear space, since ⓐ $\{f(x) = 0\} \in P$,

ⓑ if $f, g \in P$, then so is $f+g$ (why?)

ⓒ if $f \in P$, then so is kf, $\forall k \in \mathbb{R}$.

② Just like P_2, P_3, etc. we can use monomials to write out basis elements. But in $\mathcal{B} = \{1, x, x^2, x^3, \dots, x^n, \dots\}$, we will need every $n \in \mathbb{N}$. (why?)

Hence P is infinite dimensional.

Note: <u>Thm</u> Let V be a linear subspace of W. Then $0 \leq \dim V \leq \dim W$

One cannot fit a larger space into a smaller one.

Q: P is a linear subspace of $F(\mathbb{R}, \mathbb{R})$ (it is closed under linear combinations). If $\dim(P) = \infty$, then $\dim(F(\mathbb{R}, \mathbb{R})) = ?$

The concepts of linear independence, linear transformation, image, kernel, rank and nullity are all very much like that in Chapter 3.

But be very careful with <u>dimension</u>!

Def ① If the image of a linear transf. T is finite dimensional, then

$$\dim(\mathrm{im}(T)) = \mathrm{rank}(T)$$

(we define the rank to be the dimension of the image of T)

② If the kernel of T is finite dim, then $\mathrm{nullity}(T) = \dim(\ker(T))$

③ If $T: V \to W$ is a linear transf. with $\dim(V)$ finite, then the rank nullity thm (Thm 3.3.7) still holds:

$$\dim(V) = \mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(\mathrm{im}(T)) + \dim(\ker(T))$$

④ If either space of $T: V \to W$ is not finite dimensional!, then we cannot use an associated matrix A for T (why not?) (Think of the dimensions of the A matrix).

Def (I) Define $C^n(X) = \{ f : X \to \mathbb{R} \mid f$ has at least n continuous derivatives $\}$

(II) Define $C^\infty(X) = \{ f : X \to \mathbb{R} \mid f$ has all derivatives $\}$

ex. The function $f(x) = \sin x \in C^\infty(\mathbb{R})$

ex. The function $g(x) = \ln x \notin C^\infty(\mathbb{R})$ why?

But $g(x) \in C^\infty((0,\infty))$.

ex. The function $h(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x > 0 \end{cases}$ is an element

of $C^0(\mathbb{R})$ the space of continuous functions

on $\mathbb{R}$. $h(x) \in C^1(\mathbb{R})$ since $h'(x) = \begin{cases} 0 & x \le 0 \\ 2x & x > 0 \end{cases}$

is continuous on $\mathbb{R}$. But $h(x) \notin C^2(\mathbb{R})$

since $h''(x) = \begin{cases} 0 & x \le 0 \\ 2 & x > 0 \end{cases}$ is not continuous.

Hence $h(x) \notin C^n(\mathbb{R})$ $\forall n > 1$.

ex. $F(\mathbb{R}, \mathbb{R})$ is the same space as $C^0(\mathbb{R})$.

Consider the transformation

$$D : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$

defined by $D(f) = f'$, the first derivative (this is called the <u>derivative</u> <u>operator</u> on $C^\infty(\mathbb{R})$).

Q: Is D a linear ~~transf~~ transformation.

Note: $C^\infty(\mathbb{R})$ <u>is</u> a linear space. Check this!

A: Yes, since D is closed under linear combinations:

$$D(f+g) = (f+g)' \underset{\substack{\uparrow \\ \text{sum rule} \\ \text{for derivatives}}}{=} f' + g' = Df + Dg \text{ , and}$$

$$D(kf) = (kf)' \underset{\substack{\uparrow \\ \text{constant multiple} \\ \text{rule for derivatives}}}{=} kf' = kDf.$$

Hence D is a linear transformation.

Q: What is $\ker(D)$?

A: $\ker(D) = \{ f \in C^\infty(\mathbb{R}) \mid f'(x) = 0 \}$.

$= \{ f \in C^\infty(\mathbb{R}) \mid f(x) = c, \ c \in \mathbb{R} \}$.

Here $\ker(D)$ is a linear subspace of $C^\infty(\mathbb{R})$ and every element in $\ker(D)$ is a multiple of $1 = f(x)$. Hence a basis for $\ker(D)$ is $\{ f(x) = 1 \}$ and ~~so~~ $\dim(\ker(D)) = 1$.

Q: What is $\operatorname{im}(D)$?

A: The image of D is the set of all functions that can be derivatives of other functions. That is, the set of all functions that have an antiderivative in $C^\infty(\mathbb{R})$.

To answer this, we appeal to the Fundamental Thm of Calculus (FTC)

Let $f \in C^\infty(\mathbb{R})$ be ANY function.
Since it is differentiable, it is continuous on all of $\mathbb{R}$ (Thm from Calculus 1).

Hence the new function

$$F(x) = \int_0^x f(t)\,dt$$

exists on all of $\mathbb{R}$ (the integral converges).

By the FTC, $F(x)$ is also differentiable on all of $\mathbb{R}$, and $F'(x) = f \in C^\infty(\mathbb{R})$.

But then $F(x) \in C^\infty(\mathbb{R})$ (why?).

Thus every $f \in C^\infty(\mathbb{R})$ is in $\mathrm{Im}(D)$ and $\mathrm{Im}(D)$ is onto, and $\mathrm{Im}(D) = C^\infty(\mathbb{R})$.

Hence $\dim(\mathrm{Im}(D)) = \infty$.

Remark Strange: if $T: V \to V$ is a linear transformation from V to itself, then if $\dim(\ker(T)) > 0$, it cannot be the case that ~~[illegible]~~ T is onto (that $\mathrm{im}(T) = V$). This is precisely the rank-nullity thm in Chapter 3.

But in the previous example, $\dim(\ker(D)) = 1 > 0$ but D is still onto. This is because things behave a bit differently in infinite dimensions.

Def. An invertible linear transformation is called an isomorphism. If $T: V \to W$ is an isomorphism, we say V and W are isomorphic

Note: We can consider them the same space.

ex. All coordinate changes are isomorphisms.