Class 16: 10/9/13

The ideas of span, linear independence, basis, coordinates are all the same in this new abstract notion of linear space and subspace. Also dimension.

<u>ex.</u> Find a basis for $\mathbb{R}^{2\times 3}$ and determine its dim.

<u>Strategy</u>: Like with vectors, look for a way to write each element as a basis element.

<u>Solution</u>: Any $A \in \mathbb{R}^{2\times 3}$ is $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $a,b,c,d,e,f \in \mathbb{R}$.

Here,
$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_{11}} + b\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_{12}} + c\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{e_{13}}$$
$$+ d\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{e_{21}} + e\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{e_{22}} + f\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{e_{23}}.$$

Hence
$$\text{span}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \mathbb{R}^{2\times 3}.$$
$(e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23})$

But are any of these redundant?

For example, if e_{12} is redundant then

$$e_{12} = c_1 e_{11} + c_2 e_{13} + c_3 e_{21} + c_4 e_{22} + c_5 e_{23}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & c_2 \\ c_3 & c_4 & c_5 \end{bmatrix}$$ ~~which~~ are not possible.

Here a basis for $\mathbb{R}^{2\times 3}$ is $(e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23})$.
and $\dim(\mathbb{R}^{2\times 3}) = 6$.

Q: Can you show $\dim(\mathbb{R}^{n\times m}) = nm$?

ex. What is the dimension of the set of all polynomial of deg ≤ 3?

Solution: A generic element of P_3 is

$$p(x) = a + bx + cx^2 + dx^3$$
$$= a\cdot 1 + b\cdot x + c\cdot x^2 + d\cdot x^3.$$

We view the monomials $1, x, x^2, x^3$ as basis elements.
noting that $\text{span}(1, x, x^2, x^3) = P_3$
and none are redundant (we cannot write any as a <u>linear</u> combination of the others ($x^2 = x\cdot x$ is not a linear combination).

$\dim(P_3) = 4$.

Q: Can you show P_n is a linear space $\forall n \in \mathbb{N}$ and $\dim(P_n) = n+1$?

Some questions form Chapters 1, 2 and 3

Richard Brown

Mathematics Department

October 9, 2013

More than one answer is possible here

A span is

1. a basis for a vector space.
2. a finite set of vectors.
3. an infinite set of vectors.
4. a linear subspace.
5. a set of all linear combinations of a set of vectors.

More than one answer is possible here

For $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$,

1. $\mathrm{im}(T) \subset \mathbb{R}^m$.
2. $\mathrm{im}(T) \subset \mathbb{R}^n$.
3. $\ker(T) \subset \mathbb{R}^m$.
4. $\ker(T) \subset \mathbb{R}^n$.

More than one answer is possible here

A basis of an n-dimensional vector space V is

1. any finite set of vectors in V.
2. an infinite set of vectors in V.
3. The span of a set of vectors in V.
4. any linearly independent set of vectors in V.
5. any linearly independent set of vectors in V that span V.

More than one answer is possible here

For $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(\mathbf{x}) = \mathbf{Ax}$, $\mathrm{im}(A) =$

1. all solutions to $\mathbf{Ax} = \mathbf{b}$, $\forall \mathbf{b} \in \mathbb{R}^n$.
2. all solutions to $\mathbf{Ax} = \mathbf{0}$.
3. all $\mathbf{b} \in \mathbb{R}^n$ where $\mathbf{Ax} = \mathbf{b}$ is consistent.
4. all points in $\mathbb{R}^m$ mapped to a particular $\mathbf{b} \in \mathbb{R}^n$.

True or False

1. The column vectors of any 5×4 matrix must be linearly dependent.
2. If $\mathbf{A}$ is an invertible $n \times n$ matrix, then the kernels of $\mathbf{A}$ and $\mathbf{A}^{-1}$ must be equal.
3. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ span $\mathbb{R}^4$, then n must be equal to 4.
4. The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.
5. If $\mathbf{A}^2 = \mathbf{I}_n$, then $\mathbf{A}$ must be invertible.
6. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$ is a linear transformation.
7. if $\mathbf{AB} = \mathbf{I}_n$ for two matrices $\mathbf{A}$ and $\mathbf{B}$, the $\mathbf{B}$ must be the inverse of $\mathbf{A}$.
8. There exists a 3×4 matrix with rank 4.
9. A linear system with fewer unknowns than equations must have either an infinite number of solutions or no solutions.
10. A matrix $\mathbf{E}$ is in reduced-row echelon form. If we remove any single row, the resulting matrix will still be in reduced-row echelon form.