Class 15: 10/7/13

Chapter 4: Linear Spaces

The notion of linearity is not simply an $\mathbb{R}^n$ idea. Spaces and functions can be linear in an abstract sense, as long as they behave in certain ways.

Def A function from any space to any other space, as long as one can add pts together and multiply by scalars in each one, is linear if

$$f(\underbrace{c_1x_1 + c_2x_2}_{\text{addition in sending space}}) = \underbrace{c_1 f(x_1) + c_2 f(x_2)}_{\text{add. in receiving space.}}$$

$\forall x_1, x_2$ in the space and $c_1, c_2 \in \mathbb{R}$.

Def A linear space V is any set endowed with addition and scalar multiplication (if $f, g \in V$, then $f+g \in V$, and if $f \in V$ and $k \in \mathbb{R}$, then $kf \in V$)

That satisfies

① $(f+g)+h = f+(g+h)$, $f+g = g+f$

② $\exists$ a neutral element $n \in V$, where $f+n = f$ $\forall f \in V$ (basically this is the additive identity, and denoted $0 \in V$)

③ Every $f \in V$ has an additive inverse $(-f) \in V$ where $f + (-f) = 0$.

④ $k(f+g) = kf + kg$, $(c+k)f = cf + kf$, $c(kf) = (ck)f$, $1f = f$.

<u>Notes</u> Ⓐ Seriously, notice the notation. The pts in V are <u>NOT</u> vectors anymore (defined as $n \times 1$ matrices). They are simply pts in a space we call linear. Vectors are now only 1 example. This is why you do not see the vector notation any more.

Ⓑ The neutral element $0 \in V$ will look different in different spaces.

Ⓒ In a nutshell, V is linear if whenever $f, g \in V$, $c_1 f + c_2 g \in V$, i.e. V is closed under linear combos.

ex1 Very important linear space

Def The space of all real-valued functions on $\mathbb{R}$, denoted $F(\mathbb{R}, \mathbb{R})$ is a linear space, where we define addition by

$$(f+g)(x) = f(x) + g(x),$$ and

scalar multiplication by $(kf)(x) = k\,f(x)$

Note: The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0 \quad \forall x \in \mathbb{R}$ is the neutral element.

ex2 Given regular addition and scalar mult. on matrices, define

$$\mathbb{R}^{n \times m} = \{\text{all matrices } A_{n\times m}\}$$

is a linear space, with neutral element

$$0_{n\times m} = \underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_{m} \Bigg\} n$$

ex3 The set of all linear equations

$$a_1 x_1 + \cdots + a_n x_n = a_{n+1}$$

with $a_1, \cdots, a_{n+1} \in \mathbb{R}$ constants, form a linear space.

Q: What is addition and scalar multiplication?

A: Think of this as one equation in a system $A\vec{x} = \vec{b}$. Then add and scalar mult are just elementary row operations.

Q: What is the neutral element?
what is the inverse of a typical element?

ex4 The set $\mathbb{C} = \{ a + \sqrt{-1}\, b \mid a, b \in \mathbb{R} \}$ is a linear space called the complex plane.

What are addition? scalar multiplication? Neutral element?

Def W is a linear subspace of a linear space V if (same as Chapter 3)

- $0 \in W \subset V$,
- W is closed under linear combinations.

ex 5 (#10 pg 170) Show P_2 the set of all polynomials of degree ≤ 2. (of the form $f(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$) is a linear subspace of $F(\mathbb{R}, \mathbb{R})$.

Soln Since addition of polynomials is just addition of the coefficients of each monomial, it should be clear that P_2 is closed under addition of functions. And since $(kf)(x) = ka + kbx + kcx^2$, this is also true of scalar multiplication. And since we can write $f(x) = 0 = 0 + 0x + 0x^2$, the neutral element of $F(\mathbb{R}, \mathbb{R})$ is also in P_2.

Write out $(c_1 f + c_2 g)(x)$ in terms of each function f, g to see that it is a subspace.

ex6 Show that the space of all differentiable functions is a subspace of $F(\mathbb{R}, \mathbb{R})$

This is example 11, pg 170.

<u>ex 7</u> How about the space of all non-differentiable functions? Call it N_D

This is actually not a linear subspace since

① The neutral element $f(x)=0$ is actually differentiable and hence not in this space, and

② If $f(x) = 3|x| + x$, $g(x) = -3|x| + 1$ then ~~neither~~ both are in N_D. But

$$(f+g)(x) = f(x) + g(x) = 3|x| + x + (-3|x| + 1)$$
$$= x + 1$$

is not in N_D. N_D is not closed under addition.

N_D is not a linear subspace of $F(\mathbb{R}, \mathbb{R})$.