Some facts: Class 12: 10/2/13

Note: The dim. of the ker of a matrix is also called the/its nullity.

Let A be an $n \times m$ matrix:

Thm 3.3.6 $\dim(\operatorname{Im} A) = \operatorname{rank} A$.

\# lin indep columns (span columns A)

Thm. 3.3.7 Fundam. thm. of lin. alg:

$\dim(\ker A) + \dim(\operatorname{Im} A) = m$ or

(nullity of A) + (rank of A) = m.

why?: Domain of $T(\vec{x}) = A\vec{x}$ is in $\mathbb{R}^m$

Some dimensions get killed off ($\ker A$)

The rest is sent to the image: $m - \dim \ker A = \dim(\operatorname{Im} A)$

Thm 3.3.9 The vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ form a basis for $\mathbb{R}^n$

$\Rightarrow A = [\vec{v}_1 | \dots | \vec{v}_n]$ is invertible.

why? $\ker A = \{\vec{0}\}$. Use thm 3.3.6 & 7.

§ 3.4. Coordinates & Change of Basis

Recall: A basis of a vector space V is a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ s.t.:

a) $\vec{v}_1, \dots, \vec{v}_k$ are lin. indep.

b) $\vec{v}_1, \dots, \vec{v}_k$ span V.

- Consider: $\mathcal{A} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ -(the std basis)

  $\mathcal{B} = \left\{ \vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$

Are these two sets basis for $\mathbb{R}^2$?

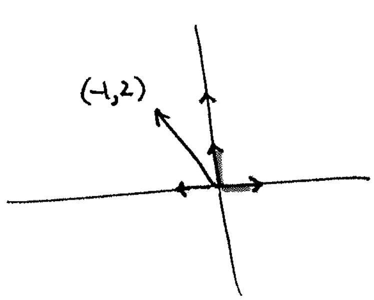
- Express the vector $\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as a lin. combinat. of the basis vectors of $\mathcal{A}$ & then of $\mathcal{B}$. What's the relation between the results?

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)\vec{v}_1 + 2\vec{v}_2$$

the coordinates of $\vec{x}$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 2\vec{w}_1 + 3\vec{w}_2$$

the $\mathcal{B}$-coord of $\vec{x}$.



$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{A}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

<u>Definition</u>: Let $\mathcal{B}$ be a basis for the vectorspace V. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$. Then every $\vec{x} \in V$ can be written uniquely as:

$$\vec{x} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k \quad \text{for some } c_1, \dots, c_k \in \mathbb{R}.$$

the scalars $c_1, \dots, c_k$ are called the $\mathcal{B}$-coordinates of $\vec{x}$ and the vector $\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ is called the $\mathcal{B}$-coord. vector of $\vec{x}$.

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}.$$

Let's look at our initial example (again):

$$\mathcal{B} = \left\{ \vec{\omega}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{\omega}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2\vec{\omega}_1 + 3\vec{\omega}_2 \quad \& \text{ thus } [x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{Let } S = [\vec{\omega}_1 \mid \vec{\omega}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

<u>note</u>: $$S[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{\omega}_1 + 3\vec{\omega}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = x$$

$$\boxed{S[\vec{x}]_{\mathcal{B}} = \vec{x}}$$

<u>Definition/Prop:</u> Given a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ ~~then~~ there is a matrix $S = [\vec{v}_1 | \dots | \vec{v}_k]$ such that $S[\vec{x}]_{\mathcal{B}} = \vec{x}$. Furthermore $[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$. S^{-1} is called the <u>change of basis matrix.</u> (why does S^{-1} exist?)

Recall: (Matrix assoc. to a lin. transformation)

Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a lin. transf. Then $\exists\, A$ st. $T(\vec{x}) = A\vec{x}$ for $\forall\, \vec{x} \in \mathbb{R}^m$. Then:

$$A = \big[\, T(e_1) \,|\, \dots \,|\, T(e_m) \,\big], \text{ where } e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{the } i^{th} \text{ row}$$

A is the matrix assoc. to T wrt the std. basis.

Definition:

Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a lin. transf, and let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a basis for $\mathbb{R}^m$. Denote by B the matrix associated to T wrt the basis $\mathcal{B}$. Then:

$$[T(\vec{x})]_{\mathcal{B}} = B\,[\vec{x}]_{\mathcal{B}}.$$

We can construct $\mathcal{B}$ column by column similarly to the case of the std basis:

$$B = \big[\, [T(\vec{v}_1)]_{\mathcal{B}} \,|\, \dots \,|\, [T(\vec{v}_m)]_{\mathcal{B}} \,\big].$$

Example: (when a different basis can be appreciated).

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that projects a vector orthogonally onto the line L spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find the matrix of the transformation T wrt the basis $\mathcal{B} = \Big\{ \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{= \vec{v}_1}, \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{= \vec{v}_2} \Big\}$.

Solution: $[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$; $B = ?$

$$B = \left[[T(\vec{v_1})]_{\mathcal{B}} \,\middle|\, [T(\vec{v_2})]_{\mathcal{B}} \right] \quad \text{by definition.}$$

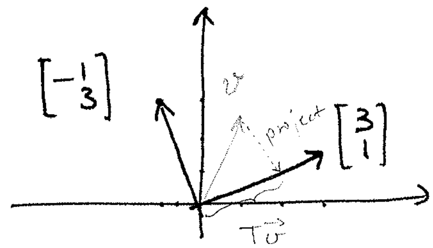
$$= \left[\left(T\begin{bmatrix}3\\1\end{bmatrix}\right)_{\mathcal{B}} \,\middle|\, \left(T\begin{bmatrix}-1\\3\end{bmatrix}\right)_{\mathcal{B}} \right]$$

- Step 1:

compute $T(\vec{v_i})$ $\forall i$

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\3\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$



- Step 2:

find the $\mathcal{B}$-coord vect. for each $T(v_i)$

$$\begin{bmatrix}3\\1\end{bmatrix} = T\begin{bmatrix}3\\1\end{bmatrix} = 1.\begin{bmatrix}3\\1\end{bmatrix} + 0\begin{bmatrix}-1\\3\end{bmatrix} = 1\vec{v_1} + 0\vec{v_2} \Rightarrow$$

$$\left(T\begin{bmatrix}3\\1\end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = T\begin{bmatrix}-1\\3\end{bmatrix} = 0\begin{bmatrix}3\\1\end{bmatrix} + 0\begin{bmatrix}-1\\3\end{bmatrix} = 0\vec{v_1} + 0\vec{v_2}$$

$$\left(T\begin{bmatrix}-1\\3\end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix}0\\0\end{bmatrix}.$$

- Step 3:

you have a matrix B

$$B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

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* We saw that there is a relation between the vector $\vec{x}$ in the std basis & $[\vec{x}]_{\mathcal{B}}$ (the vector wrt basis $\mathcal{B}$ ie. in $\mathcal{B}$-coordinates), namely $S[\vec{x}]_{\mathcal{B}} = \vec{x}$. What's the relation between the matrices A & B (where $T(\vec{x}) = A$ & $(T[\vec{x}])_{\mathcal{B}} = B[x]_{\mathcal{B}}$

Proposition: Let T be a linear transformation &
$T(\vec{x}) = A\vec{x}$ ie. A is the matrix assoc. to T wrt the std basis

$[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$ ie. B is the matrix assoc. to T wrt some basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$

Then $\underline{A = SBS^{-1}}$, where $S = [\vec{v}_1 | \dots | \vec{v}_k]$.

Proof: Let $T(\vec{x}) = \vec{\omega}$

we know $S[\vec{\omega}]_{\mathcal{B}} = \vec{\omega}$.

$$\Rightarrow \underbrace{S[\underbrace{T(\vec{x})}_{\vec{\omega}}]_{\mathcal{B}}}_{B[\vec{x}]_{\mathcal{B}}} = \underbrace{\underbrace{T(\vec{x})}_{\vec{\omega}}}_{A\vec{x}}$$

$$\Rightarrow SB[\vec{x}]_{\mathcal{B}} = A\vec{x}$$

but $\vec{x} = S[\vec{x}]_{\mathcal{B}}$

$$\Rightarrow SB[\vec{x}]_{\mathcal{B}} = AS[\vec{x}]_{\mathcal{B}}$$

this is true $\forall$ $\vec{x}$ resp $[\vec{x}]_{\mathcal{B}}$ $\Rightarrow$

$$\underline{SB = AS}.$$ ▨

Example: Use the prop. above to find the matrix of the lin. transf. $T : \mathbb{R}^2 \to \mathbb{R}^2$ from before wrt the std basis, if you know that $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, where B is the matrix wrt the basis $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

Solution: $A = SBS^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \right) =$

$$= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$

the same as: $\frac{1}{1^2+3^2} \begin{bmatrix} 3^2 & 3.1 \\ 1.3 & 1^2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$ (using the proj. f-a)

Summary:

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \\ {[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"'] \arrow[ur, phantom, "\circlearrowleft"] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"'] \end{tikzcd}$$

this is called a commut. diagram.

Def: Let A & B be two matrices. Then A & B are called <u>similar</u> if exists an invertible matrix S such that $AS = SB$.

Remark: The matrices corresponding to the same lin. transformation but computed wrt two diff. basis are similar.

Thm
Similarity is an equivalence relation; ie. it's reflexive, symmetric & transitive.

That is: if we denote by $A \sim B$ "A similar to B" then:

$A \sim A$

$A \sim B \Rightarrow B \sim A$

$A \sim B \ \& \ B \sim C \Rightarrow A \sim C$.

<u>Proof</u>: 1) $A \sim A$ because $A = I^{-1} A I$

2) $A \sim B \Rightarrow B \quad A$:

$$AS = SB \Rightarrow BS^{-1} = S^{-1}A \quad \text{call } S^{-1} = S' \Rightarrow$$

$$BS' = S'A$$

3) $A \sim B \ \& \ B \sim C \Rightarrow A \sim C$:

If $\exists S$ st. $A = S^{-1}BS$ & $\exists T$ st $B = T^{-1}CT \Rightarrow$

$$A = S^{-1}(T^{-1}CT)S$$
$$= \underbrace{(TS)^{-1}}_{M^{-1}} C \underbrace{TS}_{M}$$

$$A = M^{-1}CM \Rightarrow C \sim A.$$

∎

<u>Remark</u>: Applying elem. row operations to a matrix is just a change of basis.