Diagonalizability of a matrix is very important!

If A is diagonalizable, then

(a) It has an eigenbasis (a coordinate system where in that system of coords, A is diagonal).

(b) Algebraic and geometric multiplicities of all eigenvals are equal. (One can "see" geometrically the effects on $\mathbb{R}^n$)

(c) On each eigenspace, transformations are simple to describe (a scaling!).

However, in geometry, more important than having a basis for $\mathbb{R}^n$ is having an <u>orthonormal</u> <u>basis</u>. Why?

① Each basis vector is orthogonal to <u>all</u> of the others.

② Each basis vector has length 1.

Thus, like the standard basis, all directions are perpendicular to each other, and all measuring sticks in each direction has unit length.

Q: When does $A_{n\times n}$ have an eigenbasis which is <u>orthonormal</u>?

(When does there exist an <u>orthogonal</u> <u>matrix</u> S, where $S^{-1}AS$ is diagonal?).

<u>Def</u> $A_{n\times n}$ is <u>orthogonally</u> <u>diagonalizable</u> when $\exists$ an orthogonal S so that $S^{-1}AS$ is diagonal.

<u>Note:</u> (I) If such an S is orthogonal (cols are orthonormal) then $S^{-1} = S^T$ (Thm 5.3.7, $S^TS = I_n$).

Hence if S exists, then $S^{-1}AS = \underline{S^TAS}$ <u>is diagonal</u>.

(II) If $S^{-1}AS = D$, where D is diagonal, then

$$A = SDS^{-1} = SDS^T \quad (\text{why?})$$

But then $A^T = (SDS^T)^T = (S^T)^T D^T S^T = SDS^T = A$.

Conclusion?

$A_{n\times n}$ is symmetric here. What does this mean?

Thm 8.1.1 (Spectral Thm)

$A_{n\times n}$ is orthogonally diagonalizable iff A is symmetric.

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example 2 (pg 386) is a good one.

ex (Problem 8.1.3) Find an orthonormal basis for $A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$, and use it to diagonalize A if possible.

Strategy: Find eigenvalues and eigenvectors of A. Use to construct an orthonormal basis.

Solution: Char. eqn is $\begin{vmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$, $(6-\lambda)(3-\lambda) - 4 = 0$

or $18 - 9\lambda + \lambda^2 - 4 = \lambda^2 - 9\lambda + 14 = 0 = (\lambda-7)(\lambda-2)$

This is solved by $\lambda_1 = 2$, $\lambda_2 = 7$.

For $\lambda = 2$, we solve $A\vec{v} = 2\vec{v}$ for $\vec{v}$:

$$\left.\begin{aligned} 6v_1 + 2v_2 &= 2v_1 \\ 2v_1 + 3v_2 &= 2v_2 \end{aligned}\right\} 2v_1 = -v_2.$$ choose $v_1 = 1$ then $v_2 = -2$.

and $E_2 = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$. Call $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda_2 = 7$, system $A\vec{v} = 7\vec{v}$ is

$$\left.\begin{aligned} 6v_1 + 2v_2 &= 7v_1 \\ 2v_1 + 3v_2 &= 7v_2 \end{aligned}\right\} \quad v_1 = 2v_2 \qquad \begin{aligned} &\text{choose } v_2 = 1 \\ &\text{then } v_1 = 2. \end{aligned}$$

and $E_7 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. Call $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Note: $\vec{v}_1 \cdot \vec{v}_2 = 0$ hence $\vec{v}_1$ & $\vec{v}_2$ are orthogonal.

We normalize: $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$.

$\vec{u}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

Then $S = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_1 \\ | & | \end{bmatrix} = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $S^T = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\dfrac{1}{\sqrt{5}} = S^{-1}$

and $S^T A S = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$ do this calculation!.

Notice that in the previous example, the eigenvectors $\vec{v}_1$ & $\vec{v}_2$ were already perpendicular. This was no coincidence!

Thm 8.1.2 For $A_{n\times n}$ symmetric, if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors, respectively of $\lambda_1 \neq \lambda_2$ then $\vec{v}_1$ is orthogonal to $\vec{v}_2$ always!!
(proof in book)

Caution: Not necessarily so if $\lambda_1 = \lambda_2$ and both $\vec{v}_1$ and $\vec{v}_2$ are in the same eigenspace!!

Thm 8.1.3 A symmetric $A_{n\times n}$ always has n eigenvalues, counted with alg multiplicities.

(proof in book)

Orthogonal diagonalization of a symmetric $A_{n\times n}$.

(I) Find all eigenvalues, and a basis for each eigenspace.

(II) Using Gram-Schmidt, construct an orthonormal basis for each eigenspace
(Only need to normalize each eigenvector when λ has algebraic mult 1)

(III) Put all resulting normalized orthogonal eigenvectors together as colums of orthogonal S. Then $S^{-1}AS = S^{T}AS$ is diagonal.