Class 34: Nov 22, 2013

Some facts about considering complex numbers as eigenvalues of real-entried matrices....

(I) By Chapter 5, the complex plane $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$, with the isomorphism

$$z = a + ib \longmapsto \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$$

Hence we can simultaneously think about the algebraic nature of $z \in \mathbb{C}$ and its geometric interpretation as a vector in $\mathbb{R}^2$.

(II) If we allow complex roots to polynomials, then the Fundamental Theorem of Algebra (Thm 7.5.2) says that every degree-n polynomial $p(x)$ has n-roots counted with algebraic multiplicities.

Some facts (cont'd.)

III) If $\lambda = a+ib$ is a complex root of a characteristic polynomial of a matrix A, then so is its complex conjugate, $\bar{\lambda} = a-ib$.

Complex roots of char. polynomials of A ALWAYS come in pairs. One way to see this??

If $A_{n\times n}$ has real entries, then trace(A) and det(A) are real. But then if $\lambda = a+ib$ is one root, then another must be the unique number $\bar{\lambda} = a-ib$ so that

$$\text{trace}(A) = \lambda_1 + \cdots + \lambda_i = a+ib + \lambda_{i+1} = a-ib + \cdots + \lambda_n$$

since $(a+ib)+(a-ib) = 2a \in \mathbb{R}$, AND

$$\det(A) = \lambda_1 \cdot \cdots \cdot \lambda_i = a+ib \cdot \lambda_{i+1} = a-ib \cdot \cdots \cdot \lambda_n$$

since $(a+ib)(a-ib) = a^2+b^2 \in \mathbb{R}$.

III cont'd. If A had a complex eigenvalue without its conjugate, then trace(A) or det(A) (or both) would not be real. This would not make sense.

IV If $\lambda = a + ib$ is a complex root of char. poly of A, one can "solve" ~~the~~ the eigenvector equation <u>ONLY</u> for a complex eigenvector: $A\vec{v} = \lambda\vec{v}$, if $\lambda = a + ib$, $a, b \in \mathbb{R}$, then $\vec{v} = \vec{u} + i\vec{w}$, $\vec{u}, \vec{w} \in \mathbb{R}^2$.

Note: if $\vec{u} + i\vec{w}$ solve $A\vec{v} = (a+ib)\vec{v}$ for $\vec{v}$,

then $\vec{u} - i\vec{w}$ solves $A\vec{v} = (a - ib)\vec{v}$.

(You should show this explicitly!)

(V) In the 2x2 case, we can say alot!

All matrices with a complex eigenvalue, $\lambda = a+ib$ will ONLY have complex eigenvalues, and $\lambda = a \pm ib$.

Thus, there will be no invariant eigendirections in $\mathbb{R}^2$. The only transformations that have no invariant directions (directions where $A\vec{v} = \lambda\vec{v}$) are rotations, or rotations combined with scalings.

Thm 7.5.3 If $A_{2\times 2}$ has real entries, with eigenvalues $\lambda = a+ib$, then for $\vec{u} + i\vec{w}$ a complex solution to $A\vec{v} = \lambda\vec{v}$, we have a matrix S so that

$$S = \begin{bmatrix} | & | \\ \vec{w} & \vec{u} \\ | & | \end{bmatrix}, \text{ and } S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Notes on Thm 7.5.3

(A) The form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ should look familiar!
From Chapter 2, a rotation of $\mathbb{R}^2$ is a
linear map $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$; $R_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}$
where $a = \cos\theta$, $b = \sin\theta$.

A rotation combined with a scaling is a
linear map $R_r : \mathbb{R}^2 \to \mathbb{R}^2$,

$$R_r(\vec{x}) = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} \vec{x} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}$$

where $a = r\cos\theta$, $b = r\sin\theta$

Conversion to polar coordinates allows one to
explicitly "see" the rotation.

Hence, given any $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $r = \sqrt{a^2+b^2}$, we can
"separate" the scaling part from the
rotational part.

Notes cont'd.

(B) In fact, rotations $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$,

$$R_\theta(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}$$

never have real eigenvalues (well, okay, for $\theta = 0$, or π, or 2π, they do, but outside of these they don't).

Hence they do not have eigenspaces.

Hence ~~then~~ R_θ does not have an eigenbasis.

Hence S does not exist which can diagonalize R_θ!

The best we can hope for in the case of $A_{2\times 2}$ with complex eigenvalues is that we can write it in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ via the S in the theorem.

Back to our examples

① For $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we found $\lambda = \pm i$, and via how it acted on vectors, we concluded it was a pure rotation. B is already in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, with $a = 0$, $b = 1$. As a rotation, we can calculate θ as solution to

$$a = 0 = \cos\theta, \quad b = 1 = \sin\theta$$

The unique solution (in $[0, 2\pi)$) is $\theta = \frac{\pi}{2}$.

$\boxed{B \text{ is a rotation by } \frac{\pi}{2} \text{ in } \mathbb{R}^2}$

② For $C = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$, we found $\lambda = \frac{\sqrt{3}}{2} \pm \frac{1}{2} i$ as its eigenvalues. Hence $a = \frac{\sqrt{3}}{2}$, $b = \frac{1}{2}$, and since $r = \sqrt{a^2+b^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$

C is again a pure rotation, with

$$a = \frac{\sqrt{3}}{2} = \cos\theta, \quad b = \frac{1}{2} = \sin\theta$$

$\boxed{C \text{ is a rotation by } \frac{\pi}{6} \text{ in } \mathbb{R}^2}$

(3) For $D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\lambda = a + ib = 1 \pm i$

Here $r = \sqrt{a^2+b^2} = \sqrt{(1)^2+(1)^2} = \sqrt{2} > 1$.

Hence D is <u>NOT</u> just a rotation....

We can "pull out" the scaling part here:

$$D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\text{rotation}}$$

The other part is the rotation, in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a = \frac{1}{\sqrt{2}} = b$. And the unique solution to $\cos\theta = \frac{1}{\sqrt{2}} = \sin\theta$ is $\theta = \frac{\pi}{4}$.

D is a combination of a scaling by $\sqrt{2}$ and a rotation by $\frac{\pi}{4}$.

Again here is the picture.