Class 33: Nov 20, 2013

What conclusions can we draw from the last example of the last lecture?

Def Given λ an eigenvalue of $A_{n\times n}$, the ~~[illegible]~~ geometric multiplicity of λ is the dimension of the linear space E_λ.

Some Notes

① In the last example, the algebraic multiplicity of $\lambda=1$ was 2, while the geometric multiplicity, $\dim(E_1)=1$.

② Since we cannot produce an eigenbasis for $A=\begin{bmatrix}1&1&1\\0&0&1\\0&0&1\end{bmatrix}$ (we cannot produce 3 linearly independent eigenvectors in total), we cannot produce the matrix S, whose columns are eigenvectors, and which diagonalizes A. Hence A is NOT diagonalizable!!

Some Notes (cont'd.)

③ Let $g_i = \dim(E_{\lambda_i})$ for each distinct eigenvalue λ_i of $A_{n\times n}$, and $g = \sum g_i$.

<u>Thm 7.3.3</u> $g = n$ iff A is diagonalizable.

<u>Thm 7.3.4</u> If all eigenvalues of $A_{n\times n}$ are real and distinct, so $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, then A is diagonalizable.

④ It is <u>ALWAYS</u> the case that for λ an eigenvalue of $A_{n\times n}$,

algebraic multiplicity of λ $\geq$ geometric mult. of λ.

Think about this.

⑤ If A is similar to B, then

(i) char. poly of A = char. poly of B.

(ii) A and B have same eigenvalues! (not necessarily same eigenvectors. See below).

(iii) $\det(A) = \det(B)$ and $\operatorname{trace}(A) = \operatorname{trace}(B)$.

<u>example</u>

Is $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 2 \\ 1 & 6 & 1 \end{bmatrix}$ similar to $B = \begin{bmatrix} 1 & 1 & 4 \\ 6 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$?

One could attempt to produce an S so that $S^{-1}AS = B$. However, the answer is no, since $\operatorname{trace}(A) = 7 \neq 8 = \operatorname{trace}(B)$.

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(6) Steps for diagonalizing a matrix?

(a) Find all eigenvalues and eigenvectors.

(b) If not enough eigenvectors, matrix is not diagonalizable.

(c) If enough, then form S with columns the eigenbasis.

(d) Compute $S^{-1}AS = B$ diagonal.

ex. Diagonalize $A = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix}$.

Strategy: Follow previous steps.

Solution: Previously, we found the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, $\lambda_3 = 0$, and eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Since there are enough eigenvectors for E_5 (and E_0), we can form S and diagonalize A:

$$S = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: The order of the eigenvectors in S determines the order of the eigenvalues in $S^{-1}AS$: The entry of $(S^{-1}AS)_{ii} = \lambda_i$ and the corresponding column of S is $\vec{v}_i$ where λ_i and v_i are eigenvalue/eigenvector pairs.

Special Note:

For $A = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix}$, the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, $\lambda_3 = 0$

have eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

For the diagonal matrix $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

the eigenvalues are the same $\lambda_1 = 5$, $\lambda_2 = 5$, $\lambda_3 = 0$.

But the eigenvectors of B are now

$$\vec{v}_1 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Q: Why are the eigenvectors different?

A: Because the matrix S is a change-of-basis matrix. Thus, in the "new coordinates" on $\mathbb{R}^3$, the matrix A is the diagonal matrix B.

And since the coordinates have changed, the eigenvectors in the new coordinates look different. However, the eigenvalues have not changed!

<u>New example</u> Let $A = \begin{bmatrix} 3 & -5 \\ 1 & 1 \end{bmatrix}$. Find the eigenvalues.

<u>Solution:</u> The char. eqn is $\lambda^2 - tr(A)\lambda + det(A) = 0$

or $\lambda^2 - 4\lambda + 8 = 0$. This has no real solutions, since by quad. formula (for $a\lambda^2 + b\lambda + c = 0$)

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 32}}{2}$$

and the discriminant (under the radical) is < 0.

Hence there are no real eigenvalues of A.

Note: One could write solutions to the char. eqn. using complex numbers. Here
$\sqrt{16-32} = \sqrt{-16} = \sqrt{-1} \cdot \sqrt{16} = 4i$, for $i = \sqrt{-1}$.

Then $\lambda = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$

But then solutions to $A\vec{v} = \lambda\vec{v}$ would also be complex vectors which would not make sense since A acts on $\mathbb{R}^2$.

So since there are no real eigenvalues for A
there are no real eigenspaces for A.
Hence the change-of-basis matrix S which
would diagonalize doesn't exist.
Hence A is not diagonalizable!

Q: How to study this new situation?

Note: Also, since there are no real eigenvalues
and eigenvectors, there will not be
unique directions preserved by A
($A\vec{v} = \lambda\vec{v}$ means that certain vectors $\vec{v}$
never change direction when acted on by
A. They only grow or shrink in size, and
maybe reflect). This is a clue as
to what is going on here!!

Let's look at a few easier examples

Let $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ w/ char. eqn $\lambda^2 - 0\lambda + 1 = 0$

$C = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ w/ char. eqn $\lambda^2 - \sqrt{3}\,\lambda + 1 = 0$

$D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ w/ char. eqn $\lambda^2 - 2\lambda + 2 = 0$.

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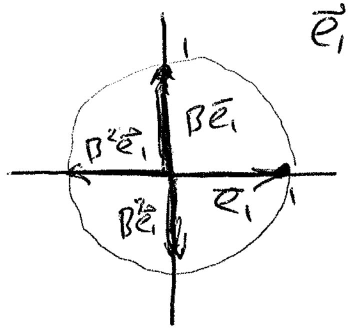
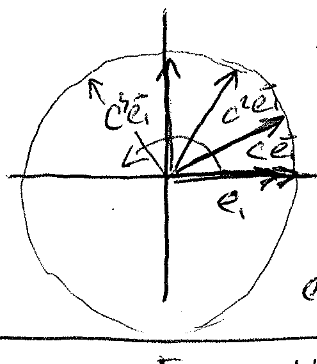
effect of B on $\mathbb{R}^2$



$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{B} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{B} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \xmapsto{B} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \xmapsto{B} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \cdots$

Here $\lambda = \frac{0 \pm \sqrt{0-4}}{2} = \pm i$ as complex numbers

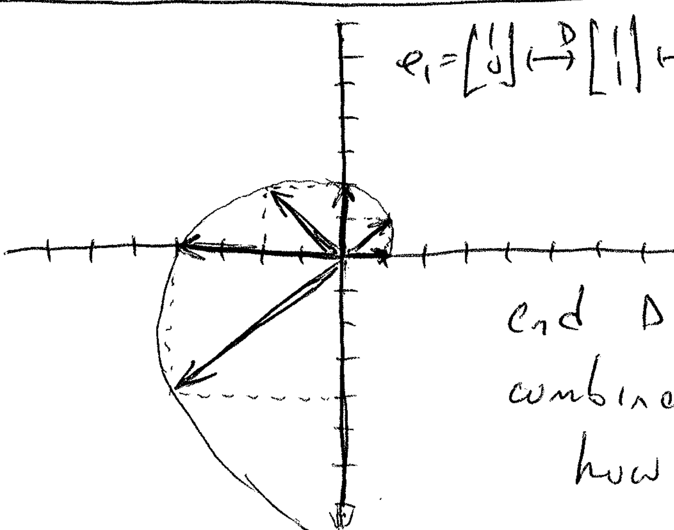
and B looks like a rotation by $\frac{\pi}{2}$.



$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{C} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 1/2 \end{bmatrix} \xmapsto{C} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \xmapsto{C} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \cdots$

Here $\lambda = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3}}{2} \pm \frac{i}{2}$ as complex numbers.

and C looks like a rotation of $\mathbb{R}^2$ by $\frac{\pi}{6}$.



$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{D} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xmapsto{D} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \xmapsto{D} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \xmapsto{D} \begin{bmatrix} -4 \\ 0 \end{bmatrix} \xmapsto{D} \begin{bmatrix} -4 \\ -4 \end{bmatrix} \longmapsto \cdots$

Here $\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

and D looks like a rotation by $\frac{\pi}{4}$ combined with a scaling by how much? (Hint: $\sqrt{2}$).