Class 32: Nov. 18, 2013

Note: In the previous example, we found that the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$ were $\lambda_1 = 3$ and $\lambda_2 = -2$, with representative eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. These 2 eigenvectors, being independent, form a basis for $\mathbb{R}^2$ $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ which is an eigenbasis for A.

In the $\mathcal{B}$-coordinates, A is diagonal, with matrix $B = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. The change-of-basis matrix is $S = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, so that

$$S^{-1} A S = B.$$

This relationship between A, B (eigenvalues are along the diagonal), and S (eigenvectors are the columns and in the same positions as their eigenvalues, is <u>No</u> <u>coincidence</u>!!

<u>ex</u> (Example 1, pg 339, suitably adapted)

Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix}$. Describe the eigenspaces.

<u>Strategy</u>: Calculate eigenvalues and eigenvectors and use the latter to describe the eigenspaces.

<u>Solution</u>: We use a cofactor expansion along the first row of $\begin{vmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 1 & -2 & 4-\lambda \end{vmatrix} = \det(A - \lambda I_3)$

to calculate the determinant:

$$\begin{vmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 1 & -2 & 4-\lambda \end{vmatrix} = (-1)^{1+1}(5-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} + (-1)^{1+2}\,0 + (-1)^{1+3}\,0$$

$$= (5-\lambda)\left((1-\lambda)(4-\lambda) - (-2)^2\right)$$

$$= -\lambda^3 + 10\lambda^2 - 25\lambda$$

Set this to 0 and solve:

$$-\lambda^3 + 10\lambda^2 - 25\lambda = 0 = -\lambda(\lambda-5)^2$$

Hence the eigenvalues are $\lambda_1 = \lambda_2 = 5$, with algebraic multiplicity 2, and $\lambda_3 = 0$.

To calculate the eigenvectors, first choose $\lambda_3 = 0$. Then $A\vec{v} = \lambda\vec{v} = 0\vec{v} = \vec{0}$ implies that the 0-eigenspace is just the kernel of A:

$$A\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 5v_1 \qquad\qquad &= 0 \\ 2v_1 + v_2 - 2v_3 &= 0 \\ v_1 - 2v_2 + 4v_3 &= 0 \end{aligned}$$

Note: Automatically, we know $v_1 = 0$, and with $v_1 = 0$, we get

$$\left.\begin{aligned} v_2 &= 2v_3 \\ -2v_2 &= -4v_3 \end{aligned}\right\} \text{same equation}$$

(Remember, this will always be the case.)

Choose $v_3 = 1$, and $v_2 = 2$, so that $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of $\lambda_3 = 0$, and

$$E_0 = \text{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

Now choose $\lambda_1 = \lambda_2 = 5$. The system $A\vec{v} = 5\vec{v}$ is

Ex. (cont'd)

$$A\vec{v} = 5\vec{v} \Rightarrow \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5v_1 \\ 5v_2 \\ 5v_3 \end{bmatrix} \Rightarrow \begin{array}{l} 5v_1 = 5v_1 \\ 2v_1 + v_2 - 2v_3 = 5v_2 \\ v_1 - 2v_2 + 4v_3 = 5v_3 \end{array}$$

$$\Rightarrow \begin{array}{l} v_1 = v_1 \\ \left.\begin{array}{l} 2v_1 - 4v_2 - 2v_3 = 0 \\ v_1 - 2v_2 - v_3 = 0 \end{array}\right\} \text{same equation} \end{array}$$

Here we "need" 2 independent solutions:

(a) choose $v_1 = 1$, $v_3 = 1$. Then $v_2 = 0$, and

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is one eigenvector.

(b) To make the 2nd choice automatically independent of the 1st, choose $v_2 = 1$, $v_3 = 0$. Then $v_1 = 2$ (why is $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ independent of $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$?)

Then $E_5 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Conclusions we can draw here?

(I) 0 is a perfectly acceptable eigenvalue for a matrix A. In fact, when $\ker(A)$ is a positive dimensional subspace, it is the 0-eigenspace of A. In these directions, all ~~eigen~~ vectors get mapped to the 0-multiple of themselves.

(II) If an eigenvalue has ~~multiplicity~~ algebraic multiplicity > 1, what can be said about the dimension of the eigenspace? Can it be > 1? Is it necessarily > 1?

very important questions!

(III) The transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$ is a composition of a projection onto a 2-dim subspace $V = E_5$, and a scaling by 5. Can you see this??

ex. (Example 3 pg 340)

Find the eigenspaces of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Strategy: Same as previous example.

Solution Eigenvalues are solutions to

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 = (1-\lambda)^2(-\lambda), \text{ and are } \lambda = 0, 1.$$

and we will call them $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$.

To find the eigenvectors, choose $\lambda_1 = 0$. Here the system $A\vec{v} = 0\vec{v} = \vec{0}$ is

$$\left.\begin{aligned} v_1 + v_2 + v_3 &= 0 \\ v_3 &= 0 \\ v_3 &= 0 \end{aligned}\right\} \quad v_1 = -v_2 \text{ is the only equation.}$$

Choose $v_1 = 1$, so $v_2 = -1$, $v_3 = 0$, and

$$E_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Example 3 (cont'd.)

Now choose $\lambda_1 = \lambda_2 = 1$. Here $A\vec{v} = 1\vec{v} = \vec{v}$ is

$$\left.\begin{aligned} v_1 + v_2 + v_3 &= v_1 \\ v_3 &= v_2 \\ v_3 &= v_3 \end{aligned}\right\} \quad \begin{aligned} &v_1 = \text{anything} \\ &v_2 = -v_3 \\ &v_2 = v_3 \end{aligned}$$

The only solution to the last 2 equations is $v_2 = v_3 = 0$. And since $v_1 = \text{anything}$, we get

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Here, even though $\lambda = 1$ is an eigenvalue of algebraic multiplicity 2, the eigenspace E_1 is only 1-dimensional.

Conclusions?

(I) The answer to the last part of the item (II) in the other conclusions is "No, it is not necessarily the case that $\dim(E_\lambda)$ = algebraic multiplicity of λ.

Conclusions (cont'd.)

II There aren't enough eigenvectors around to "fill out" all of $\mathbb{R}^3$ in this case. This means there is not an ~~eigenspace~~ eigenbasis for A.

This means that A is not diagonalizable!

We will explore this next class.