Class 31: Nov 15, 2013

We start with another example:

ex. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{bmatrix}$. Here we know

$\det A = 3(2)(-2) = -12$ since the matrix is upper triangular. Find the eigenvalues.

Strategy: Compute $\det(A - \lambda I_3) = 0$, and solve.

Solution: $\det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

Note that $A - \lambda I_3$ is still upper triangular, and we are simply computing the det of it:

$$\det(A - \lambda I_3) = (3-\lambda)(2-\lambda)(-2-\lambda) = 0$$

which is solved by (it is already factored)

$$\lambda = 3, 2, -2.$$ $\blacksquare$

The eigenvalues of an upper (or lower) triangular matrix are the entries of the main diagonal.

Notes ① Given $A_{n\times n}$, $\det(A-\lambda I_n)$ is just an nth degree polynomial in the only variable λ. There can be only at most n roots of such a polynomial (why?)

② The polynomial $\det(A-\lambda I_n)$ is called the characteristic polynomial of A.

③ The equation $\det(A-\lambda I_n)=0$ is called the characteristic equation of A.

③ Special case: $n=2$. Let $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$.

Here $\det(A-\lambda I_2) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda\end{bmatrix} = 0$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= ad - a\lambda - d\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

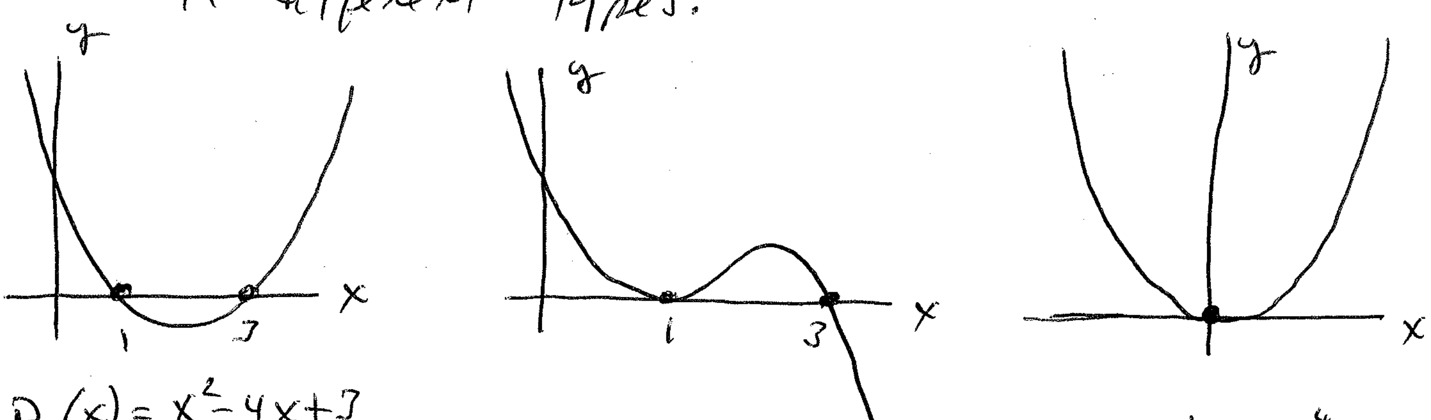
$$= \lambda^2 - tr(A)\lambda + \det(A) = 0$$

Is the characteristic equation of A. it ALWAYS has this form.

(4) There are similar patterns for $A_{n\times n}$, $n>2$, but they are more complicated and not so useful to remember.

(5) The book denotes the characteristic polynomial of $A_{n\times n}$ by $f_A(\lambda)$.

Recall the roots of a polynomial can come in different types.



$p_2(x) = x^2 - 4x + 3$
$= (x-3)(x-1)$

$p_3(x) = -x^3 + 5x^2 - 7x + 3$
$= -(x-1)^2(x-3)$

$p_4(x) = x^4$

<u>Def</u> Given a polynomial $p(x)$, if λ is a root, where $p(x) = (x-\lambda)^k q(x)$,

and $q(\lambda) \neq 0$, then we say λ has <u>algebraic multiplicity</u> k.

Note ① if $k=1$, λ is called a simple root
if $k>1$, λ is a multiple root. Sometimes we would say λ is a double root if $k=2$, or a triple root if $k=3$.

② Fact: $A_{n\times n}$ has at most n real eigenvalues counted with multiplicities.

③ Fact: if n is odd, then $p(x)$ must have at least one real root. (why?)

④ Denote the n-eigenvalues of $A_{n\times n}$, if they exist $\lambda_1, \cdots, \lambda_n$.

Then $$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n = \prod_{i=1}^{n} \lambda_i$$

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^{n} \lambda_i$$

Special Note

If completely factorable, for $A_{n\times n}$,

$$\det(A-\lambda I_n) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = 0$$

and eigenvalues are $\lambda_1, \cdots, \lambda_n$, all real.

Special Note (cont'd.)

However, for polynomials of degree $n > 4$, there are no general methods for easy root finding. Hence finding eigenvalues is HARD.

There are methods for certain types, however, and computer algebra systems to help.

But in general this is not an easy problem.

Back to original eqns: $A\vec{v} = \lambda\vec{v}$, or $(A - \lambda I_n)\vec{v} = \vec{0}$.

Only when λ is an eigenvalue of A is
$\det(A - \lambda I_n) = 0$.

Here then $\ker(A - \lambda I_n) \neq \{\vec{0}\}$ (nullity is positive).

In this case, the linear subspace of $\mathbb{R}^n$ corresponding to λ is the λ-eigenspace E_λ, where $E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v}\}$.

Note: ANY $\vec{v} \in E_\lambda$ is a λ-eigenvector of $A_{n\times n}$ (an eigenvector corresponding to λ).

ex. Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$. We previously found $\lambda = 3, -2$ as the 2 eigenvalues of A. Describe the eigenspaces $E_3, E_{-2} \subset \mathbb{R}^2$.

Strategy: Solve $A\vec{v} = 3\vec{v}$, $A\vec{v} = -2\vec{v}$ for all $\vec{v}$.

Solution: Choose $\lambda = 3$. Use either $A\vec{v} = 3\vec{v}$, or $(A - 3I_2)\vec{v} = \vec{0}$.

Here $A\vec{v} = 3\vec{v} \Rightarrow \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is the system

$$\left.\begin{aligned} 4v_1 - 2v_2 &= 3v_1 \\ 3v_1 - 3v_2 &= 3v_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} v_1 &= 2v_2 \\ v_1 &= 2v_2 \end{aligned}\right\} \text{ same equation! coincidence??}$$

Any vector satisfying $v_1 = 2v_2$ is a 3-eigenvector of A: choose $v_2 = 1$, $v_1 = 2$.

Then $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $E_3 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = -2$, the system is $(A - (-2)I_2)\vec{v} = \vec{0}$, and

$$\begin{bmatrix} 4+2 & -2 \\ 3 & -3+2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} 6v_1 - 2v_2 &= 0 \\ 3v_1 - v_2 &= 0 \end{aligned}\right\} \text{ both are just } 3v_1 = v_2$$

choose $v_1 = 1$, $v_2 = 3$, so that $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $E_{-2} = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.