Class 30: Nov 13, 2013

Diagonal Matrices

Consider the three systems:

$$(\mathrm{I})\quad \begin{aligned} x_1+2x_2+3x_3&=y_1\\ 4x_1+5x_2+6x_3&=y_2\\ 7x_1+8x_2+9x_3&=y_3 \end{aligned} \qquad (\mathrm{II})\quad \begin{aligned} x_1+2x_2+3x_3&=y_1\\ 4x_2+5x_3&=y_2\\ 6x_3&=y_3 \end{aligned} \qquad (\mathrm{III})\quad \begin{aligned} x_1&=y_1\\ 2x_2&=y_2\\ 3x_3&=y_3 \end{aligned}$$

Every equation in system (I) includes every x variable, so that system is rather hard to solve. Such a system is called (fully) coupled, since untangling the equations takes work.

In contrast, system (III) is (fully) uncoupled, as each equation only involves one unknown. System can be solved individually.

In (II), system is partially uncoupled and solving one equation leads to easy solutions of the others.

Given a linear transformation

$$T: \mathbb{R}^n \to \mathbb{R}^n, \quad T(\vec{x}) = A\vec{x}$$

it would be good to know if $\exists$ a basis $\mathfrak{B}$ for $\mathbb{R}^n$ where, with respect to $\mathfrak{B}$, the matrix A look like a matrix B which is <u>diagonal</u> (only nonzero entries on the main diagonal, like the coefficient matrix in System (III) above), <u>BEFORE</u> trying to solve.

<u>Def.</u> If such a matrix B for a basis $\mathfrak{B}$ exists, the matrix A is called <u>diagonalizable</u>

Note: If such a B exists, then B is similar to A, and via the change of basis matrix S, (for $\mathfrak{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$, $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$),

$$S^{-1}AS = B.$$

ex. Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$. For the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

A diagonalizes to $B = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. Show this.

Strategy: Use $\mathcal{B}$ to construct S, and reconstruct the eqn. $S^{-1}AS = B$.

Solution: By Thm 4.3.4 (or thm 3.4.4, or Def 3.4.1), the change of basis matrix for $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is $S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$.

Here $S = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ then, and $S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$.

Then
$$S^{-1}AS = \underbrace{\frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 9 & -3 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 15 & 0 \\ 0 & -10 \end{bmatrix} =$$
$$= \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$
$$= B. \quad \blacksquare$$

Now let $A_{2\times 2}$ be diagonalizable, and $\mathcal{B}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ be the diagonal matrix using the basis $\mathcal{B}=\{\vec{v}_1, \vec{v}_2\}$, so that the change-of-basis matrix $S=\begin{bmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{bmatrix}$, and

$$AS = S\mathcal{B}$$

This equation is vital, since:

On the left: $AS = A\begin{bmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{bmatrix} = \begin{bmatrix}| & |\\ A\vec{v}_1 & A\vec{v}_2\\ | & |\end{bmatrix}$ $\left(\begin{smallmatrix}\text{Thm}\\ 2.3.2\end{smallmatrix}\right)$

right: $S\mathcal{B} = \begin{bmatrix}| & |\\ \vec{v}_1 & \vec{v}_2\\ | & |\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}| & |\\ \lambda\vec{v}_1 & \lambda\vec{v}_2\\ | & |\end{bmatrix}$.

These are equal, so we know

$$A\vec{v}_1 = \lambda_1\vec{v}_1 \text{ and } A\vec{v}_2 = \lambda_2\vec{v}_2$$

<u>Notes</u> ① As long as A is diagonalizable, this works.

② Works in n-dimensions the same way, and one gets n-equations: $A\vec{v}_i = \lambda_i\vec{v}_i$.

Notes (cont'd)

(3) ANY nonzero vector $\vec{v}$ that satisfies $A\vec{v} = \lambda\vec{v}$ for some choice of λ is called an eigenvector of A. Then the $\lambda \in \mathbb{R}$ is called an eigenvalue of A.

(4) Eigenvalues for $A_{n\times n}$ are rare, and there are at most n of them for any given A.

Any nonzero multiple of an eigenvector is also an eigenvector

(5) If $\vec{v}$ is an eigenvector of A, then so is $k\vec{v}$, for all $k \in \mathbb{R}$, since if $A\vec{v} = \lambda\vec{v}$, then $A(k\vec{v}) = k(A\vec{v}) = k(\lambda\vec{v}) = \lambda(k\vec{v})$.

(6) A basis $\mathcal{B} = \{\vec{v}_1 \ldots \vec{v}_n\}$ of $\mathbb{R}^n$ is called an eigenbasis for A (or T) if each $\vec{v}_i$ is an eigenvector of A.

Thm $A_{n \times n}$ is diagonalizable iff $\exists$ an eigenbasis for A.

The eigenvalues and eigenvectors of $A_{n \times n}$ are extremely important geometric properties of $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$.

Q: Can one calculate these properties without diagonalizing the matrix directly?

A: Yes, and the place to do so is $A\cancel{\vec{x}} = \lambda\vec{v}$.

To find eigenvalues and eigenvectors of $A_{n \times n}$, one must "solve" $A\vec{v} = \lambda\vec{v}$ for both $\vec{v}$ and λ.

This is a system, with n-equations, but with $n+1$ unknowns: each entry in $\vec{v}$ and a λ.

The trick is to solve for one at a time.

Rewrite $A\vec{v} = \lambda\vec{v}$ as $A\vec{v} - \lambda\vec{v} = \vec{0}$

(careful here: the right hand side is the zero <u>vector</u>!)

One would like to "factor out" the common $\vec{v}$ here, but $A\vec{v} - \lambda\vec{v} \neq (A - \lambda)\vec{v}$ because it does not make sense (why not?).

Instead, first write $A\vec{v} - \lambda\vec{v} = A\vec{v} - \lambda I_n \vec{v} = \vec{0}$ since $\vec{v} = I_n \vec{v}$.

Then $A\vec{v} - \lambda\vec{v} = \boxed{(A - \lambda I_n)\vec{v} = \vec{0}}$

Now $(A - \lambda I_n)$ is an $n \times n$ matrix, with $\vec{v}$ and $\vec{0}$ two n-vectors.

Here, solving for $\vec{v}$ is like looking for nontrivial solutions to a system of equations with the right hand side equal to $\vec{0}$.

Note: Think of $\boxed{(A - \lambda I_n)\vec{v} = \vec{0}}$ as

the equation that is helping you solve for the kernel of a linear transformation whose matrix is $A - \lambda I_n$.

Q: When would $(A - \lambda I_n)\vec{v} = \vec{0}$ have NON-TRIVIAL solutions $\vec{v}$?

A: Precisely (and only) when $A - \lambda I_n$ is a NON-INVERTIBLE matrix!

This happens only when $\det(A - \lambda I_n) = 0$ (why?)

Here, now $\det(A - \lambda I_n) = 0$ is an equation (only one equation) with only one unknown λ. Any solution to this will be an eigenvalue of A!

ex Go back to $A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$. Find its eigenvalues.

<u>Strategy</u>: Solve $\det(A - \lambda I_2) = 0$ for all possible values of λ.

<u>Solution</u>: $\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$$= \det\left(\begin{bmatrix} 4-\lambda & -2 \\ 3 & -3-\lambda \end{bmatrix}\right)$$

$$= (4-\lambda)(-3-\lambda) - (-6) = 0$$

$$= -12 - 4\lambda + 3\lambda + \lambda^2 + 6 = 0$$

$$= \lambda^2 - \lambda - 6 = 0$$

$$= (\lambda - 3)(\lambda + 2) = 0$$

The two solutions, $\lambda = 3$, $\lambda = -2$ are the two eigenvalues of A. ∎