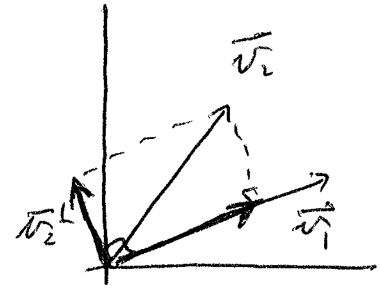
Class 29; Nov. 11, 2013 I

Last class we developed a geometric description of the determinant of a 2×2 matrix:

Given $A_{2\times2} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$, $|\det A| = \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\|$,

where $\vec{v}_2 = v_2^{\parallel} + v_2^{\perp}$, and

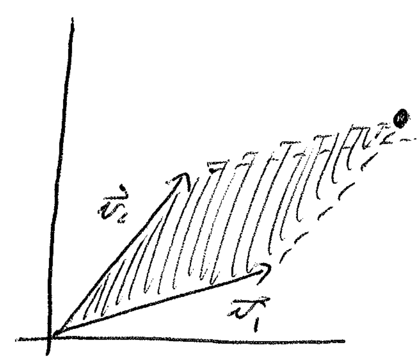
$\vec{v}_2^{\parallel} = \text{proj}_{\text{span}(\vec{v}_1)} \vec{v}_2$



But what does the quantity $|\det A|$ actually mean?

Q: What is the area of the parallelogram at right

A: Like a rectangle, it is the base $\cdot$ height. But the measurements need to be orthogonal: Choose $\vec{v}_1$ as the base, with length $\|\vec{v}_1\|$. Then height is $\vec{v}_2^{\perp}$, with length $\|\vec{v}_2^{\perp}\|$.



We then set $|\det(A)| = \|\vec{v}_1\|\,\|\vec{v}_2^{\perp}\|$

$$= \left\{\begin{array}{l}\text{area of parallelogram} \\ \text{with sides } \vec{v}_1, \vec{v}_2\end{array}\right.$$

It turns out, this is much more general than even this result:

In general, any matrix $A_{n\times n} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ with linearly independent columns (so that $\det A \neq 0$) has a QR factorization where Q is orthonormal (so $|\det(Q)| = 1$) and an upper triangular R, whose main diagonal entries are

$$r_{11} = \|\vec{v}_1\|, \quad r_{ii} = \|\vec{v}_i^{\perp}\|$$

via the Gram-Schmidt Process. Hence

$$|\det(A)| = |\det(QR)| = |\det(Q)|\,|\det(R)|$$

$$= \underbrace{1}_{\substack{Q \text{ is} \\ \text{orthonormal.}}} \cdot \underbrace{\|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdot \ldots \cdot \|\vec{v}_n^{\perp}\|}_{\substack{\text{main diagonal} \\ \text{elements of } R.}}$$

But this is precisely the "volume" of the n-dimensional parallelpiped whose sides are formed by the $\vec{v}_1, \ldots, \vec{v}_n$ vectors.

<u>Thm</u> Let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then the m-volume of the m-dimensional parallelpiped in $\mathbb{R}^n$ defined by $\vec{v}_1, \ldots, \vec{v}_m$ is

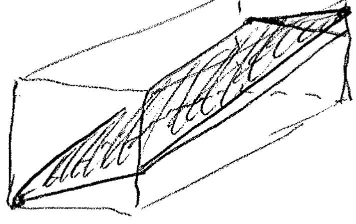
$$\sqrt{\det(A^T A)},$$

where $A_{n \times m} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$.

And, in the case where $m = n$, the volume is

$$|\det(A)|.$$

Note: When $m < n$, we are looking for the volume of a smaller dimensional parallelpiped in $\mathbb{R}^n$, like the length of a line segment in $\mathbb{R}^2$ or a parallelogram slice through a cube in $\mathbb{R}^3$.



ex. Find the area of the parallelogram in $\mathbb{R}^3$ formed by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ in $V = \text{span}\{\vec{v}_1, \vec{v}_2\}$.

Solution: By the theorem, the area (the 2-volume) of the parallelogram with sides $\vec{v}_1$ and $\vec{v}_2$ is given by $\sqrt{\det(A^T A)}$, where

$$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus $A^T A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$

and $\det(A^T A) = 12 - 9 = 3$, so the area is $\sqrt{3}$.

Why is this important?

- Linear transformations of $\mathbb{R}^n$ may or may not preserve volume. If they do not, then either volume grows or shrinks.

How much a linear transformation changes a given volume is called its <u>expansion factor</u>:

(I) For a lin. transf. $T: \mathbb{R}^n \to \mathbb{R}^n$, this expansion factor is the same for any set with positive volume anywhere in $\mathbb{R}^n$.

Contrast this with vector fields in Calc III and Diff. Eqns, where a measure of volume expansion of a vector field flow is given by its <u>divergence</u>:

Divergence free vector fields preserve volume. But in a nonlinear vector field, volume may be shrinking in some regions and expanding in others. Not so in a linear system.

(II) How to calculate it?

Given $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$.

Def. ~~The~~ For a given region $\Omega \subset \mathbb{R}^n$, where $0 < vol(\Omega) < \infty$, the ratio of the volume of the image of Ω to the volume of Ω is the expansion factor of T (or A).

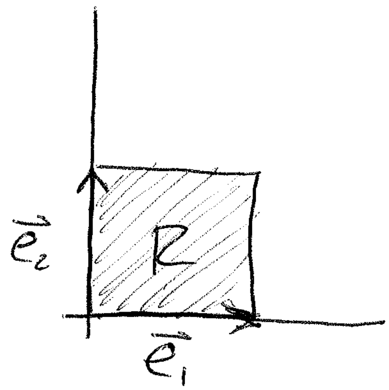
$$= \frac{vol(T(\Omega))}{vol(\Omega)}$$

Note: Since the expansion factor is the same everywhere in $\mathbb{R}^n$ and it doesn't matter what region Ω one takes, one can just start with the unit n-cube.

ex. Find the expansion factor of ~~A~~

$T:\mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{x}) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\vec{x}$ $\left(A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\right)$.

<u>Solution</u> What happens to the unit square R:

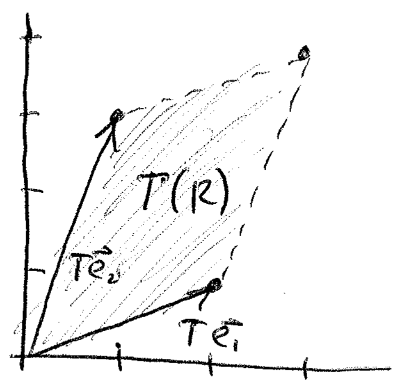


$\xrightarrow{\quad T \quad}$

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

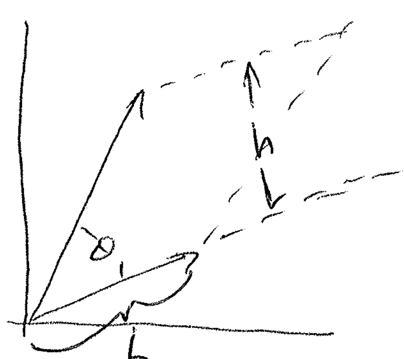


With linear transformations, polygons go to polygons, parallelograms to parallelograms, parallelpipeds to parallelpipeds, etc.
(Just find out where adjacent corners go and draw straight lines!)

Here $\text{vol}(R) = 1$.

$\text{vol}(T(R)) = bh = \|T\vec{e}_1\| h$

and $h = \|T\vec{e}_2\| \sin\theta = \|T\vec{e}_2^{\perp}\|$



Hence $\|T\vec{e}_1\| \|T\vec{e}_2^{\perp}\| = \text{vol}(T(R))$.

But by the previous work we did, given a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, we can construct the <u>matrix</u> for T (call it A) by $A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$ (remember Chapter 1).

But this is precisely $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ as before.

And $|\det A| = \|T\vec{e}_1\| \, \|T\vec{e}_2^{\perp}\| = \text{vol}(T(R))$

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5.$$

Hence the expansion factor here is 5.