Class 28: Nov 8, 2013

Exercise 6.2.1 pg 289.

Calculate $\det\begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix}$ using row reduction.

Strategy: Row reduce $\begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix}$ to upper triangular and then calculate determinant, keeping track of data.

Solution:

Method 1: First, divide row 3 by 2 to render all rows have a leading 1.

$$A = \begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix} \xrightarrow[\frac{1}{2}\text{row}3 \to \text{row}3]{} \begin{bmatrix}1&1&1\\1&3&3\\1&1&5/2\end{bmatrix}$$

Now create all 0's in 1^{st} column below top:

$$\begin{bmatrix}1&1&1\\1&3&3\\1&1&5/2\end{bmatrix} \xrightarrow[\substack{\text{row}1-\text{row}2\to\text{row}2\\ \text{row}1-\text{row}3\to\text{row}3}]{} \begin{bmatrix}1&1&1\\0&-2&-2\\0&0&-3/2\end{bmatrix} = B$$

So $\det(A) = (-1)^0 2 \det(B) = 2(1(-2)(-3/2)) = 6$

Exercise 6.2.1 pg 289 (cont'd).

Method 2: Only use step © above, adding multiples of rows to other rows with replacement.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix} \xRightarrow[\substack{\text{row1} - \text{row2} \to \text{row2} \\ 2\,\text{row1} - \text{row3} \to \text{row3}}]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -3 \end{bmatrix} = B.$$

$$\det(A) = \det(B) = 1(-2)(-3) = 6.$$

Other useful properties

Ⓐ For $A_{n\times n}$, $B_{n\times n}$, $m \in \mathbb{N}$, we have

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^m) = (\det(A))^m$$

Ⓑ For $A_{n\times n}$ and $B_{n\times n}$ similar,

$$\det(A) = \det(B) \quad (\text{why?})$$

Note: A and B are similar if $\exists S_{n\times n}$, where $AS = SB$.

But then $\det(AS) = \det(A)\det(S) = \det(S)\det(B)$

$= \det(SB)$

$\boxed{\text{Hence } \det(A) = \det(B)}$

Other useful properties (cont'd).

(c) If $A^{-1}_{n\times n}$ exists, then

$$\det(A^{-1}) = \frac{1}{\det(A)} = (\det(A))^{-1} \quad \text{(why?)}$$

Here $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$

Let $A_{n\times n}$ be an orthogonal matrix (its columns form an orthonormal basis for $\mathbb{R}^n$)

Then by Thm 5.3.7, $A^T A = I_n$.

But this means
$$\det(A^T A) = \det(I_n) = 1$$
$$\det(A^T)\det(A) = [\det(A)]^2$$

Since for any $A_{n\times n}$, $\det(A^T) = \det(A)$
(this is Thm 6.2.1)

Thus for $A_{n\times n}$ orthogonal, $\det A = \pm 1$.

<u>Def</u> A matrix $A_{n\times n}$ orthogonal where $\det(A) = 1$ is called a <u>rotation</u>.

<u>Notes</u> ① In dimension 2, this should be obvious, since only the rotations (about the origin) are orthogonal. ~~[illegible]~~
(well, also the -1 ones, though).

cont'd.

Notes ② Orthogonal matrices w/ determinant -1 are rotations composed with some sort of reflection.

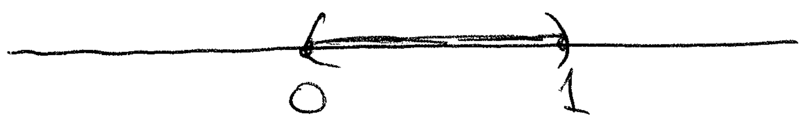
③ In dimension 3, a <u>rotation</u> is always about an axis, or 1-d subspace that doesn't change. To think about this, take the surface of a globe in $\mathbb{R}^3$ centered at the origin and rotate it. There will always be an axis that doesn't move.

④ Recall, orthogonal matrices always preserve the size of vectors. And that is a feature of rotations also.

⑤ In higher dimensions, $(n>3)$, rotations are hard to visualize and can be very complicated.

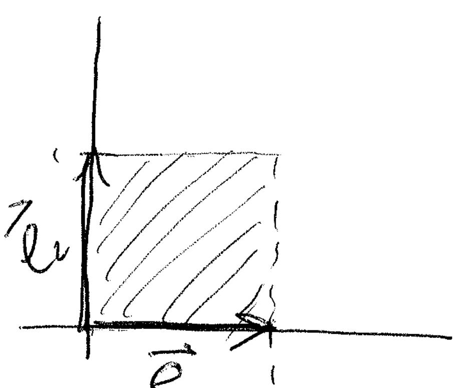
Here is a nice interpretation of the determinant: As a measure of volume in $\mathbb{R}^n$.

Volume in dimension $n=1$ is just length. And the unit interval has length 1:



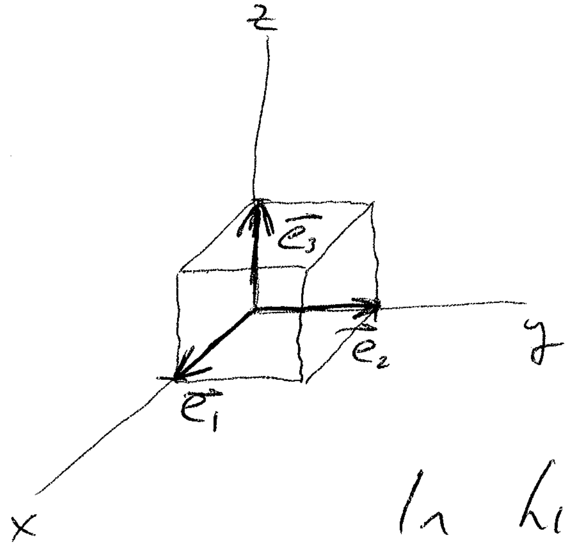
Euclidean distance is $\mathrm{dist}(x,y) = |y-x|$.

Volume in dimension $n=2$ is area, and the unit square in $\mathbb{R}^2$ has area 1:



Here in $\mathbb{R}^2$, $\|\vec{e}_1\| = \|\vec{e}_2\| = 1$, and area of square is base times height or $1 \cdot 1 = 1$.

Volume in $\mathbb{R}^3$ is what we think of volume, and the volume of the unit cube is 1:



$$\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{e}_3\| = 1$$

and volume is base·width·height $= 1\cdot 1\cdot 1 = 1$.

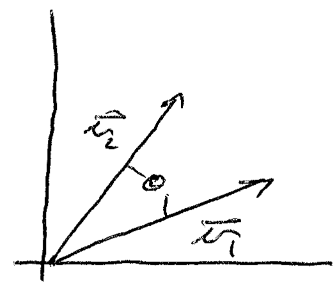
In higher dimensions, we play the exact same game, with a n-dimensional <u>hypercube</u>, an analogue of the above shapes where the edges are formed by $\vec{e}_1, \dots, \vec{e}_n$ and "opposite" edges are parallel, and whose n-dimensional volume is just the product of the lengths of the "directions".

We also call the hypercube in n-dimensions the "n-cube".

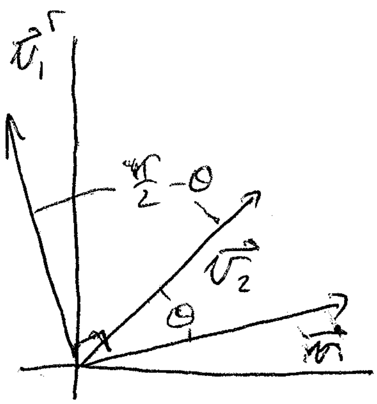
ex. ~~A~~ The unit square in $\mathbb{R}^2$ is a 2-cube.

V

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$, where $\vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$,
be a 2x2 matrix, and denote θ the angle between $\vec{v}_1$ and $\vec{v}_2$



Denote by $\vec{v}_1^{\,r}$ the rotation of $\vec{v}_1$ by the angle $\frac{\pi}{2}$ radians. Here we know



(A) $\vec{v}_1^{\,r}$ is the same length as $\vec{v}_1$, so $\|\vec{v}_1^{\,r}\| = \|\vec{v}_1\|$, and

(B) $\vec{v}_1^{\,r}$ is orthogonal to $\vec{v}_1^{\,r}$:

$$\vec{v}_1^{\,r} = \begin{bmatrix} -c \\ a \end{bmatrix} \quad \text{(check this!)}$$

Note: The book describes this in Section 2.4 (pg 94) and calls $\vec{v}_1^{\,r}$ by $\vec{v}_{rot}$. I chose the notation since I want to keep the subscript 1.

We can now write

$$\vec{v}_1{}^r \cdot \vec{v}_2 = \begin{bmatrix} -c \\ a \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = ad - bc = \det(A)$$

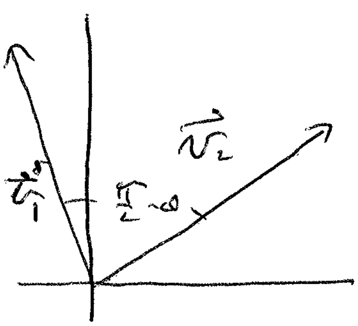
This is an interesting geometric way to write the determinant, and since for any $\vec{u}, \vec{w}$,

$$\vec{u} \cdot \vec{w} = \|\vec{u}\| \, \|\vec{w}\| \cos\theta,$$

We can say

$$\det(A) = \vec{v}_1{}^r \cdot \vec{v}_2 = \|\vec{v}_1{}^r\| \, \|\vec{v}_2\| \cos\left(\tfrac{\pi}{2} - \theta\right)$$

$$= \|\vec{v}_1\| \, \|\vec{v}_2\| \sin\theta$$



due to the facts that

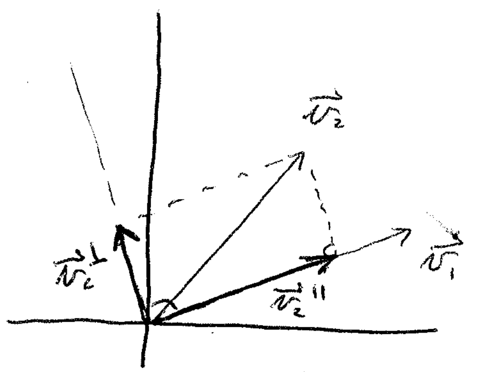
ⓐ $\|\vec{v}_1{}^r\| = \|\vec{v}_1\|$, and

ⓑ $\cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$. (check this!)

Now consider ONLY the "size" or "magnitude" of the determinant:

$$|\det A| = \|\vec{v}_1\| \, \|\vec{v}_2\| \, |\sin\theta|$$

Write $\vec{v}_2$ as the sum of $\vec{v}_2^{\,\parallel}$ a component in the direction of $\vec{v}_1$, and $\vec{v}_2^{\,\perp}$ the component orthogonal to $\vec{v}_1$:



$$\vec{v}_2 = \vec{v}_2^{\,\parallel} + \vec{v}_2^{\,\perp}.$$

Then $\|v_2^{\perp}\| = \|\vec{v}_2\|\,|\sin\theta|$ by Pythagorean Theorem.

This means that
$$|\det A| = \|\vec{v}_1\|\,\underbrace{\|\vec{v}_2\|\,|\sin\theta|}$$
$$= \|\vec{v}_1\|\,\|v_2^{\perp}\|$$

Next time, we will see what this means geometrically.