Class 27: Nov 6, 2013

Some other determinant properties

(I) Thm 6.2.1 For $A_{n\times n}$, $\det(A) = \det(A^T)$.

Note: This a priori may not be so obvious. But if you look closely at theorem 6.2.10 (last class), you will see that the ith row cofactor expansion of $A_{n\times n}$ is precisely the ith column cofactor expansion of $A^T_{n\times n}$. As they will be the same and are stated as such in Thm 6.2.10, this theorem follows.

(II) The determinant is ~~linear~~ not linear!

Q: Is the transformation

$$T : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$$

$$T(A) = \det(A)$$

a linear transformation?

II cont'd.

A: It would be if

$$T(k_1 A_1 + k_2 A_2) \overset{?}{=} k_1 T(A_1) + k_2 T(A_2).$$

So is it?

example Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Here $\det(A) = 2 - 1 = 1$.

And $\det(kA) = \det\left(\begin{bmatrix} k & k \\ k & 2k \end{bmatrix}\right) = 2k^2 - k^2 = k^2$.

So $\det(kA) = T(kA) = k^2 \det(A)$

$= k^2 T(A)$.

when A is 2×2.

A: T is not a linear transformation!

Follow up question: For $A_{n \times n}$, what would you expect for the box? (why?)

$$\det(kA) \overset{?}{=} \boxed{} \det(A).$$

II cont'd.

But determinant <u>is</u> linear in each row or column individually (we say it is multi-linear here, but that is not a part of this course):

Consider $T_{r_i}: \mathbb{R}^{1\times n} \to \mathbb{R}$, given by

$$T_{r_i}(\vec{x}) = \det \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_{i-1} & - \\ - & \vec{x} & - \\ - & \vec{v}_{i+1} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$$

where $\vec{v}_i \in \mathbb{R}^{1\times n}$ are all fixed vectors, and $\vec{x}$ is the ith row.

This is a <u>linear</u> <u>transformation</u> (show this).

And the same is true for $T_{c_i}: \mathbb{R}^n \to \mathbb{R}$

$$T_{c_i}(\vec{x}) = \det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{x} & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix}$$

Recall T is linear if $T(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1T(\vec{x}_1) + c_2T(\vec{x}_2)$

III On elementary row operations? How do determinants behave?

(a) Divide a row ~~by~~ of A by $k \in \mathbb{R}$. to create a new matrix B.

Then $\det(B) = \frac{1}{k}\det(A)$

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ex. In the 2×2 case, let $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$.
Here $\det(A) = 6-4 = 2$. Create B by dividing row 1 of A by 2: $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

$\det(B) = 3-2 = 1 = \frac{1}{2}2 = \frac{1}{2}\det(A)$.

(b) Swap row i with row j to create B: $\det(B) = -\det(A)$

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ex. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $\det(A) = 6-4 = 2$.
Create B via a swap of rows
$B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $\det(B) = 4-6 = -2 = -\det(A)$.

III cont'd.

(C) Create B by adding a multiple of one row to another and replacing one of the rows with the result:

$$\det(B) = \det(A)$$

These three operations on matrices allows us to actually calculate the determinant of $A_{n \times n}$ via a row reduction of A to a new matrix whose determinant is easier to calculate.

This presents us with another technique for determinant calculation.

Suppose we row reduce $A_{n\times n}$ to I_n, and in the process, we

(i) divided various rows by scalers $k_1, \dots, k_r \in \mathbb{R}$, and also

(ii) swapped rows s times, and

(iii) added multiples of rows to other rows and replaced....

Then $$1 = \det(\underbrace{\operatorname{rref}(A)}_{I_n}) = (-1)^s \frac{1}{k_1 \cdots k_r} \det(A),$$

or $$\det(A) = (-1)^s k_1 \cdots k_r \det(\operatorname{rref}(A))$$
$$= (-1)^s k_1 \cdots k_r \quad \text{in this case.}$$

Note: This method is much more computationally efficient than the cofactor expansion, and is typically used by computer algebra systems.

Follow up Questions

Q1: And if $rref(A) \neq I_n$. What then?

Q2: What if we only row-reduced $A_{n\times n}$ so that it is upper triangular. Is this helpful?

In both cases, we still have

$$\det(A) = k_1 \cdots k_r (-1)^s \det(rref(A))$$

or

$$\det(A) = k_1 \cdots k_r (-1)^s \det(B)$$

where B is some level of row reduction on A, involving the $k_1, \dots, k_r$ divides and s swaps.

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ex 4 in 6.2 is an excellent example

Exercise 6.2.1 pg 289.

Calculate $\det\begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix}$ using row reduction.

Strategy: Row reduce $\begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix}$ to upper triangular and then calculate determinant, keeping track of data.

Solution:

Method 1: First, divide row 3 by 2 to render all rows have a leading 1.

$$A = \begin{bmatrix}1&1&1\\1&3&3\\2&2&5\end{bmatrix} \xrightarrow[\frac{1}{2}\text{row}3 \to \text{row}3]{} \begin{bmatrix}1&1&1\\1&3&3\\1&1&5/2\end{bmatrix}$$

Now create all 0's in 1st column below top:

$$\begin{bmatrix}1&1&1\\1&3&3\\1&1&5/2\end{bmatrix} \xrightarrow[\substack{\text{row}1-\text{row}2 \to \text{row}2\\ \text{row}1-\text{row}3 \to \text{row}3}]{} \begin{bmatrix}1&1&1\\0&-2&-2\\0&0&-3/2\end{bmatrix} = B$$

So $\det(A) = (-1)^0 2 \det(B) = 2(1(-2)(-3/2)) = 6$

Exercise 6.2.1 pg 289 (cont'd).

Method 2: Only use step © above, adding multiples of rows to other rows with replacement.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix} \xrightarrow[\text{2 row1} - \text{row3} \to \text{row3}]{\text{row1} - \text{row2} \to \text{row2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -3 \end{bmatrix} = B.$$

$$\det(A) = \det(B) = 1(-2)(-3) = 6.$$

Other useful properties

Ⓐ For $A_{n\times n}$, $B_{n\times n}$, $m \in \mathbb{N}$, we have

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^m) = (\det(A))^m$$

Ⓑ For $A_{n\times n}$ and $B_{n\times n}$ similar,

$$\det(A) = \det(B) \quad (\text{why?})$$

Note: A and B are similar if $\exists S_{n\times n}$, where $AS = SB$.

But then $\det(AS) = \det(A)\det(S) = \det(S)\det(B) = \det(SB)$

$\boxed{\text{Hence } \det(A) = \det(B)}$

Other useful properties (cont'd).

(c) If $A_{n\times n}^{-1}$ exists, then

$$\det(A^{-1}) = \frac{1}{\det(A)} = (\det(A))^{-1} \quad \text{(why?)}$$

Here $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$