Class 26: Nov. 4, 2013

Chapters 6 and 7 revolve around the information found inside a matrix

First step – the determinant.

Recall the determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc$$

and that

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

A^{-1} exists iff $ad - bc = \det(A) \neq 0$.

The determinant of $A_{n \times n}$

See Mathematica output for 3x3 and 4x4 cases.

1. The common denominator of the elements of the inverse of a matrix when it exists, found via the process of reversing $A\vec{x} = \vec{b}$ to $\vec{x} = A^{-1}\vec{b}$
2. denoted in matrix notation:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \det A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

The determinant is (cont'd)

(3) non-zero precisely when A^{-1} exists.

(4) the indicator that $A\vec{x}=\vec{b}$ has a unique solution or not. When $\det A \neq 0$, there is a unique solution to $A\vec{x}=\vec{b}$. When $\det A = 0$, system either has no solutions or an infinite number.

How to define the determinant for $A_{n\times n}$?

In essence, the same way: $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

- Form the $2n\times n$ matrix $[A \,\vdots\, I_n]$.
- Row reduce it to $[I_n \,\vdots\, A^{-1}]$.

  If possible.

If one kept track of all the operations and used the entries w/o numbers, the common denominator of elements of A^{-1} would be $\det(A)$. See next page.

In[85]:=
```
Print["A22= ", MatrixForm[A22 = {{a11, a12}, {a21, a22}}]]
Print["A33= ", MatrixForm[A33 = {{a11, a12, a13}, {a21, a22, a23}, {a31, a32, a33}}]]
Print["A44= ", MatrixForm[A44 = {{a11, a12, a13, a14},
    {a21, a22, a23, a24}, {a31, a32, a33, a34}, {a41, a42, a43, a44}}]]
```

A22= $\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$

A33= $\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix}$

A44= $\begin{pmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ a31 & a32 & a33 & a34 \\ a41 & a42 & a43 & a44 \end{pmatrix}$

In[88]:=
```
MatrixForm[Inverse[A22]]
Print["The Determinant of A22 is ", Det[A22], ".  Compare with above deniminator."]
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Out[88]//MatrixForm=

$$\begin{pmatrix} \frac{a22}{-a12\,a21+a11\,a22} & -\frac{a12}{-a12\,a21+a11\,a22} \\ -\frac{a21}{-a12\,a21+a11\,a22} & \frac{a11}{-a12\,a21+a11\,a22} \end{pmatrix}$$

The Determinant of A22 is $-a12\,a21+a11\,a22$. Compare with above deniminator.

In[90]:=
```
MatrixForm[Inverse[A33]]
Print["The Determinant of A33 is ", Det[A33], ".  Compare with above deniminator."]
```

Out[90]//MatrixForm=

$$\begin{pmatrix} \frac{-a23\,a32+a22\,a33}{-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21\,a33+a11\,a22\,a33} & \frac{a13\,a32-a12\,a33}{-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21} \\ \frac{a23\,a31-a21\,a33}{-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21\,a33+a11\,a22\,a33} & \frac{-a13\,a31+a11\,a33}{-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21} \\ \frac{-a22\,a31+a21\,a32}{-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21\,a33+a11\,a22\,a33} & \frac{a12\,a31-a11\,a32}{-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21} \end{pmatrix}$$

The Determinant of A33 is
$-a13\,a22\,a31+a12\,a23\,a31+a13\,a21\,a32-a11\,a23\,a32-a12\,a21\,a33+a11\,a22\,a33$
. Compare with above deniminator.

In[92]:=
```
Print["The a11 element of A44 is ", Inverse[A44][[1, 1]]]
Print["The Determinant of A44 is ", Det[A44], ".  Compare with above deniminator."]
```

The a11 element of A44 is
$(-a24\,a33\,a42+a23\,a34\,a42+a24\,a32\,a43-a22\,a34\,a43-a23\,a32\,a44+a22\,a33\,a44)\,/$
$(a14\,a23\,a32\,a41-a13\,a24\,a32\,a41-a14\,a22\,a33\,a41+a12\,a24\,a33\,a41+a13\,a22\,a34\,a41-$
$a12\,a23\,a34\,a41-a14\,a23\,a31\,a42+a13\,a24\,a31\,a42+a14\,a21\,a33\,a42-$
$a11\,a24\,a33\,a42-a13\,a21\,a34\,a42+a11\,a23\,a34\,a42+a14\,a22\,a31\,a43-a12\,a24\,a31\,a43-$
$a14\,a21\,a32\,a43+a11\,a24\,a32\,a43+a12\,a21\,a34\,a43-a11\,a22\,a34\,a43-a13\,a22\,a31\,a44+$
$a12\,a23\,a31\,a44+a13\,a21\,a32\,a44-a11\,a23\,a32\,a44-a12\,a21\,a33\,a44+a11\,a22\,a33\,a44)$

The Determinant of A44 is
$a14\,a23\,a32\,a41-a13\,a24\,a32\,a41-a14\,a22\,a33\,a41+a12\,a24\,a33\,a41+a13\,a22\,a34\,a41-$
$a12\,a23\,a34\,a41-a14\,a23\,a31\,a42+a13\,a24\,a31\,a42+a14\,a21\,a33\,a42-a11\,a24\,a33\,a42-$
$a13\,a21\,a34\,a42+a11\,a23\,a34\,a42+a14\,a22\,a31\,a43-a12\,a24\,a31\,a43-a14\,a21\,a32\,a43+$
$a11\,a24\,a32\,a43+a12\,a21\,a34\,a43-a11\,a22\,a34\,a43-a13\,a22\,a31\,a44+$
$a12\,a23\,a31\,a44+a13\,a21\,a32\,a44-a11\,a23\,a32\,a44-a12\,a21\,a33\,a44+a11\,a22\,a33\,a44$
. Compare with above deniminator.

Note: Procedure would be ridiculously tedious!

Q: Are there other ways?

A: Yes, of course.

Section 6.1 details a combinatoric way to recreate the complicated expression for $\det(A_{n\times n})$.

We will pass on this method for a better one:

First, for 3x3 matrices?

(I) Via the cross product

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$.

Note: $\det A \neq 0$ precisely when $\vec{u}, \vec{v}, \vec{w}$ are linearly independent. (why?)

Consider $\vec{v} \times \vec{w}$, the cross product of 2 vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ (Calculus III)

In $\mathbb{R}^3$, $\vec{v}\times\vec{w}$ is a vector perpendicular to both $\vec{v}$ and $\vec{w}$, and

$$\vec{v}\times\vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

If we then take the "dot product" of the result with $\vec{u}$, this product is non zero precisely when $\vec{u}$ and $\vec{v}\times\vec{w}$ are <u>NOT</u> orthogonal:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

Compare with Mathematica Output

$$\left.\begin{aligned} &= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 \\ &\quad - u_2v_1w_3 + u_3v_1w_3 - u_3v_2w_1 \end{aligned}\right.$$

That is, this is nonzero when $\vec{u},\vec{v},\vec{w}$ are lin. indep.

(II) Sarrus' Rule - A clever way to do the same thing:

with plus signs

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

with minus signs

write this out and compare.

Both techniques are valid for $\mathbb{R}^3$.
Both fail in higher dimensions!

<u>Def.</u> Given $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, define the

$A_{23} = \begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}$.

$\updownarrow$

$\begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} & a_{14} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} & \cancel{a_{24}} \\ a_{31} & a_{32} & \cancel{a_{33}} & a_{34} \\ a_{41} & a_{42} & \cancel{a_{43}} & a_{44} \end{bmatrix}$

<u>ijth minor</u> A_{ij} (note the capital letter)

$$A_{ij} = \begin{cases} \text{the } (n-1)\times(n-1) \text{ matrix formed by} \\ \text{removing the } i\text{th row and the } j\text{th} \\ \text{column.} \end{cases}$$

<u>Thm 6.2.10</u> Given $A_{n\times n}$, the <u>Laplace expansion</u> (or cofactor expansion) along the jth column is

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

and along the ith row is

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Notes ① To find the det of $A_{n\times n}$ requires finding the determinants of n matrices of size $(n-1)\times(n-1)$, which requires each, ... recursively.

② One can choose ANY row or column for this calculation. So choose the one with the most 0's in it (why?)

③ Minor matrices are also called cofactors

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ex. Calculate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution: Choose 1^{st} row:

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35)$$

$$= -3 + 12 - 9 = 0$$

what does this mean?

<u>ex</u> Calculate $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$, $a, b, c \in \mathbb{R}$.

<u>Solution</u>: Here $\det\left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\right)$ (choose 1st col).

$$= a\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ b & 0 \end{vmatrix}$$

$= abc$, the product of main diagonal.

<u>ex</u>. Calculate $\begin{vmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{vmatrix}$, all real numbers.

<u>Solution</u>: Choose 4th row.

$$\text{determinant} = (-1)^{4+1} a_{41} A_{41} + (-1)^{4+2} a_{42} A_{42} + (-1)^{4+3} a_{43} A_{43} + (-1)^{4+4} a_{44} A_{44}$$

$$= 0 + 0 + 0 + (1)\, j \begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & h \end{vmatrix}$$

choose 1st column

$$= j\left(a\begin{vmatrix} e & f \\ 0 & h \end{vmatrix} - 0\begin{vmatrix} b & c \\ 0 & h \end{vmatrix} + 0\begin{vmatrix} b & c \\ e & f \end{vmatrix}\right)$$

$= jaeh$ = product of main diagonal.
<u>again</u>.