Another example

Def The trace of a square matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \text{is} \quad tr(A) = \sum_{i=1}^{n} a_{ii}$$

The sum of the diagonal elements

Notes ① Similar matrices have the same trace.

② For $A_{n \times n}$, $tr(A) = tr(A^T)$ (why?)

ex. On the space $\mathbb{R}^{n \times m}$, define

$$\langle A, B \rangle = tr(A^T B)$$

Show this is an inner product space.

Notes: ① For $A_{n\times m}, B_{n\times m}$, $A^T B$ is square $(m \times m)$.

② $\langle A, B \rangle = tr(A^T B) = tr((A^T B)^T) = tr(B^T A) = \langle B, A \rangle$.

We can use the inner product in a linear space V to define notions like

ⓐ length or magnitude of elements of V

ⓑ orthogonality of elements of V

ⓒ orthonormal bases,

ⓓ distance between elements in V

in the same way we do this with the dot product in $\mathbb{R}^n$.

<u>Def</u> (I) The norm (magnitude or length) of an element $f \in V$ a linear space with an inner product is

$$\|f\| = \sqrt{\langle f, f\rangle}$$

(II) Any 2 elements $f, g \in V$ an inner product space are called orthogonal or perpendicular if $\langle f, g\rangle = 0$.

Def cont'd

(III) The distance between $f, g \in V$ an inner product space is

$$\text{dist}(f,g) = \| f - g \|$$

Note: In $\mathbb{R}^n$, an inner product space using the dot product on vectors, Euclidean distance is

$$\text{dist}(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

But this is just

$$\sqrt{(\vec{x}-\vec{y}) \cdot (\vec{x}-\vec{y})} = \| \vec{x} - \vec{y} \|$$

from the geometry one sees in Calculus III.

With inner products on spaces, we can easily talk about abstract linear spaces and their elements in very concrete terms.

ex. What is the magnitude (norm, length) of the function $f(t) = t^2 - 1$ in the space $C^0([0,1])$?

Strategy: Use the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$.

Solution: Calculate directly:

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_0^1 (t^2-1)^2\,dt \right)^{1/2}$$

$$= \left(\int_0^1 (t^4 - 2t^2 + 1)\,dt \right)^{1/2} = \left(\frac{t^5}{5} - \frac{2}{3}t^3 + t \Big|_0^1 \right)^{1/2}$$

$$= \left(\frac{1}{5} - \frac{2}{3} + 1 \right)^{1/2} = \sqrt{\frac{8}{15}}.$$

ex. Show $f(t) = t$ and $g(t) = t^2$ are orthogonal on $C^0([-1,1])$.

Solution. Calculate

$$\langle f, g \rangle = \int_{-1}^1 t \cdot t^2\,dt = \int_{-1}^1 t^3\,dt = ?$$

(Hint: t^3 is an odd function).

Follow up questions to last example:

(a) Let $n \in \mathbb{N}$. Are t^n and t^{n+1} also orthogonal?

Answer: Yes since on $C^0([-1,1])$, using the inner product $\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$

$\langle t^n, t^{n+1}\rangle = \int_{-1}^{1} t^{2n+1}\,dt$. Here the integrand is an odd-function $\forall n \in \mathbb{N}$.

~~Hence~~ And $\int_{-1}^{1} (\text{odd fnc})\,dt = 0$.

(b) Does this mean that the set of functions $\mathcal{B} = \{1, t, t^2, t^3, t^4, \cdots, t^n, \cdots\}$ orthonormal?

Answer: No, for many reasons.

(1) Choose $f(t) = t^2$.

$$\|f\| = \|t^2\| = \left(\int_{-1}^{1} t^2 \cdot t^2\,dt\right)^{1/2} = \left(\frac{t^5}{5}\Big|_{-1}^{1}\right)^{1/2}$$

$$= \sqrt{\frac{2}{5}} \neq 1.$$

Hence t^2 is not a unit-length element of $\mathcal{B}$. But also, even though t and t^2 are orthogonal, AND t^2 and t^3 are orthogonal, it does not mean that t and t^3 are orthogonal. In fact (do the calculation) they are <u>NOT</u>!

For an orthonormal set, all pairs of elements must be orthogonal.

(c) Is the example still true if we change the space from $C^0([-1,1])$ to $C^0([0,1])$.

Answer: No, since t and t^2 are orthogonal in $C^0([-1,1])$, but in $C^0([0,1])$,

$$\langle t, t^2\rangle = \int_0^1 t^3\,dt = \frac{t^4}{4}\Big|_0^1 = \frac{1}{4} \neq 0.$$

Hence t and t^2 are not orthogonal in $C^0([0,1])$.

The space as well as the product matter.

ex. Given again $f(t) = t$, $g(t) = t^2$ on $C^0([-1,1])$ what is the distance between them?

<u>Solution</u>: Calculate

$$\|t - t^2\| = \left(\int_{-1}^{1} (t-t^2)^2\, dt\right)^{1/2} = \left(\int_{-1}^{1} (t^2 - 2t^3 + t^4)\, dt\right)^{1/2}$$

$$= \left(\frac{t^3}{3} - \frac{1}{2}t^4 + \frac{t^5}{5}\Bigg|_{-1}^{1}\right)^{1/2}$$

$$= \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \left(\frac{-1}{3}\right) + \left(\frac{1}{2}\right) - \left(\frac{-1}{5}\right)\right)^{1/2}$$

$$= \left(\frac{2}{3} + \frac{2}{5}\right)^{1/2} = \sqrt{\frac{16}{15}}$$

Lastly, we can discuss orthogonal projections on linear spaces V w/ an inner product in exactly the same way we did on $\mathbb{R}^n$:

Let $\mathcal{B} = \{g_1, \cdots, g_m\}$ be an orthonormal basis of a linear subspace W of an inner product space V. Then

$$\operatorname{proj}_W f = \langle g_1, f \rangle g_1 + \cdots + \langle g_m, f \rangle g_m$$

This is exactly the same formula as Section 5.1 (Thm 5.1.5).