Class 36: Dec 2, 2013

Here is an application of symmetric matrices and orthogonal diagonalization:

Consider the following function: $g: \mathbb{R}^2 \to \mathbb{R}$,

$$g(\vec{x}) = g(x_1, x_2) = 8x_1^2 - 4x_1x_2 + 5x_2^2$$

It should be obvious that this function is

(a) nonlinear, and (b) $g(0,0) = 0$.

A good vector calculus question is:

Q: Is $(0,0)$, or $\begin{bmatrix}0\\0\end{bmatrix}$ critical for g?
If so, is it an extremum?
If so, what kind?

This (These) would be a very good (set of) question(s) for applications, and are natural questions to ask of functions....

~~[illegible]~~

Def A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a quadratic form on $\mathbb{R}^n$ if it is a linear combination of monomials of the form $x_i x_j$, where $i, j = 1, \cdots, n$ and may be equal.

Note: A quadratic form on $\mathbb{R}^n$ can always be written $g(\vec{x}) = \vec{x} \cdot A\vec{x} = \vec{x}^T A \vec{x}$ for $A_{n \times n}$ a symmetric matrix.

ex. What is the matrix for the $g : \mathbb{R}^2 \to \mathbb{R}$ defined above?

Solution: Rewrite (x_1, x_2) as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$: Then

$$g(\vec{x}) = 8x_1^2 - 4x_1x_2 + 5x_2^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 8x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}$$

$$= [x_1 \ x_2] \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \vec{x}^T A_{2\times 2} \vec{x}, \text{ where } A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}. \blacksquare$$

It should be clear that if we change from one-coordinate system to another via a change of basis matrix, that the result will still look like a quadratic form:

Indeed, let S be a 2×2 change-of-basis matrix, so that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = S\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $S^{-1}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = g\left(S^{-1}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$$
$$= g\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = (S^{-1}\vec{y})^T A\, (S^{-1}\vec{y})$$
$$= \vec{y}^T (S^{-1})^T A\, S^{-1}\vec{y}$$
$$= \vec{y}^T B\, \vec{y}$$

where $B = (S^{-1})^T A\, S^{-1}$ is the new matrix.

It is straightforward to show by entry ij that B is also a symmetric matrix (do this!!)

Q: Is there a preferred coordinate system where q is easier to study?

A: By the Spectral Thm (last class: Section 8.1), there exists an orthonormal basis for A (making it orthogonally diagonalizable in the new coordinates).

Diagonalize A via an orthogonal S:

① Characteristic eqn: $\lambda^2 - 13\lambda + 36 = 0$
$= (\lambda - 9)(\lambda - 4)$

Eigenvalues of A are $\lambda_1 = 9$, $\lambda_2 = 4$.

② Eigenvectors: $\vec{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

③ Here $S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, and

$$S^{-1}AS = S^TAS = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

To switch coordinates, let $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ (so that the new coordinates are (c_1, c_2), or $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2$)

Here $g(\vec{x}) = g(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{x} \cdot A\vec{x}$

$$= (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot A(c_1\vec{v}_1 + c_2\vec{v}_2)$$

$A\vec{v} = \lambda\vec{v} \implies$

$$= (c_1\vec{v}_1 + c_2\vec{v}_2) \cdot (c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2)$$

$\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal basis $\rightarrow$

$$= c_1^2\lambda_1(\vec{v}_1 \cdot \vec{v}_1) + \cancel{2c_1c_2\lambda_1\lambda_2(\vec{v}_1 \cdot \vec{v}_2)}^{\,0} + c_2^2\lambda_2(\vec{v}_2 \cdot \vec{v}_2)$$

$$= c_1^2\lambda_1 + c_2^2\lambda_2$$

Hence, with $\lambda_1 = 9$, $\lambda_2 = 4$, $g: \mathbb{R}^2 \to \mathbb{R}$,

$$g(\vec{c}) = 9c_1^2 + 4c_2^2 = \vec{c} \cdot D\vec{c}$$

where $D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$ is the orthogonal diagonalization of A.

Q: Now, answer the questions: Is $(0,0)$ critical? Is it an extremum?

Def Given a quadratic form on $\mathbb{R}^n$, $g(\vec{x}) = \vec{x} \cdot A\vec{x}$, for a symmetric matrix A, we say $g(\bar{x})$ is

(I) positive definite if $g(\vec{x}) > 0$ for all nonzero $\bar{x} \in \mathbb{R}^n$,

(II) positive semidefinite if $g(\bar{x}) \geq 0$ for all nonzero $\bar{x} \in \mathbb{R}^n$.

(III) negative definite and negative semi definite if comparatives are changed to $<$ and $\leq$ respectively.

(IV) indefinite if none of the above.

Notes: (1) The diagonalized example from above $g: \mathbb{R}^2 \to \mathbb{R}$, $g(\bar{x}) = 9x_1^2 + 4x_2^2$ is positive definite. (Show this!)

Notes cont'd.

(2) Metrics (notions of distance, like Euclidean metric on $\mathbb{R}^n$) are positive definite by definition (see the definition).

We call the pair (V, g) where V is a vector space and g a quadratic form a quadratic space.

ex: ($\mathbb{R}^n$, Euclidean distance) is a quadratic space.

(3) Thm 8.2.4 A symmetric matrix is positive definite iff all of its eigenvalues are positive and semidefinite iff all of its eigenvalues are nonnegative.

Def Given a quadratic form $q(\vec{x}) = \vec{x}^T A \vec{x}$ where $A_{n\times n}$ is symmetric with n distinct eigenvalues, the eigenspaces are called the principal axes of q.

Notes: ① All eigenspaces here are 1-dim.

② All eigenspaces are orthogonal to each other (why?)

③ ex. The principal axes of

$$q: \mathbb{R}^2 \to \mathbb{R}, \quad q(\vec{x}) = \vec{x}^T \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} \vec{x}$$

are $E_9 = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$, $E_4 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

Thm 8.2.7 Let $q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2 = 1$. Define the curve C as the set of solutions in $\mathbb{R}^2$. Let λ_1, λ_2 be the eigenvalues of $A_{2\times 2}$ of q.

ⓐ if $\lambda_1, \lambda_2 > 0$, C is an ellipse

ⓑ if $\lambda_1 > 0 > \lambda_2$, C is a hyperbola.

(and if $\lambda_1 = \lambda_2 > 0$, what shape is C???)