EXAMPLE: PROBLEM 4.2.18 MEAN VALUE THEOREM

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Question 1. *Show the equation* $x^3 + e^x = 0$ *has exactly one real solution.*

Strategy: We need to do two things here. First, we show that the equation has a solution. Then we will need to show that there cannot be two solutions. We will use the Intermediate Value Theorem for the existence of a solution, and Rolle's Theorem (or the MVT) for the fact that there is only one. Our strategy will include the idea that for the function $f(x) = x^3 + e^x$, this problem is equivalent to saying that the graph of $f(x)$ has only one x-intercept.

Solution: First note that the function $f(x) = x^3 + e^x$ is differentiable on all of $\mathbb{R}$, since it is the sum of two differentiable functions, a polynomial and the exponential function. Hence it is continuous. Grabbing the two input values $x = -1$ and $x = 0$, we see that $f(-1) = (-1)^3 + e^{-1} = -1 + \frac{1}{e} < 0$, and $f(0) = (0)^3 + e^0 = 1 > 0$. Hence by the Intermediate Value Theorem, we know that there must be a number $c \in [-1, 0]$, where $f(c) = 0$. But this is our solution to the original equation. Hence a solution exists.

So call c our solution to the original equation and suppose there exists another one; Call this one d. Then either $d > c$ or $d < c$. It really won't matter for our discussion, so assume $d > c$. Then $f(x)$ is a continuous function on $[c, d]$ and differentiable on (c, d) (it is differentiable everywhere, actually). By Rolle's Theorem, there then must exist a point $r \in (c, d)$, where $f'(r) = 0$. However, we have

$$f'(x) = 3x^2 + e^x > 0$$

for all $x \in \mathbb{R}$. Thus, like our example in class, there cannot be a point $r \in (c, d)$, where $f'(r) = 0$. So our assumption that there is a second solution to the original equation is incorrect. Thus there is only one real solution to $x^3 + e^x = 0$.

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