EXAMPLE: PROBLEM 3.10.46 RELATED RATES

110.109 CALCULUS I (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Question 1. *The minute hand of a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?*

Strategy: First, we give our unknown distance the variable s. Now both the hour and minute hand are moving at constant rates. Hence the angle between them is also changing at a constant rate. Call this angle θ. The two hands and the distance between the tips form a triangle, and we can relate the lengths of the sides to the angles with the Law of Cosines (a generalization of the Pythagorean Theorem for non-right triangles. We use the Law of Cosines to relate the distance between the tips to the angle between the hands, and then differentiate with respect to time to relate the rates.

Solution: For any triangle, one can relate the lengths of the three sides to one of the angles of the triangle via the Law of Cosines $a^2 = b^2 + c^2 - 2(b)(c)\cos\theta$. In our case, let h be the hour hand and m be the minute hand. Then, at one o'clock, we can form a triangle as in the figure. Here the angle is $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ (why??), and we can use the Law of Cosines to find the distance s between the tips of the hands at that moment. Note that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so that

$$s^2 = m^2 + h^2 - 2(m)(h)\cos\theta = 8^2 + 4^2 - 2(8)(4)\cos\frac{\pi}{6} = 80 - 64\left(\frac{\sqrt{3}}{2}\right) = 80 - 32\sqrt{3}. \tag{1}$$

So $s = \sqrt{80 - 32\sqrt{3}}$ mm at precisely one o'clock. Notice that in Equation 1, the only variables that change are s and θ. If we differentiate this equation with respect to time, we get

$$\frac{d}{dt}\left[s^2 = m^2 + h^2 - 2(m)(h)\cos\theta\right], \quad \Longrightarrow \quad 2s\frac{ds}{dt} = -2(m)(h)(-\sin\theta)\frac{d\theta}{dt}$$
$$= 2(8)(4)\left(\frac{1}{2}\right)\frac{d\theta}{dt}.$$

(Make sure you get this!) Thus the quantity we are looking for is

$$\left.\frac{ds}{dt}\right|_{\theta=\frac{\pi}{6}} = \left.\left(\frac{32}{2s}\frac{d\theta}{dt}\right)\right|_{\theta=\frac{\pi}{6}} = \left(\frac{16}{\sqrt{80 - 32\sqrt{3}}}\right)\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{6}}.$$

We have everything but how the angle is changing with respect to time.

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To find this, note that the hour-hand makes one revolution every hour (we make clockwise positive movement here). Hence the hour-hand angle has a rate of change of 2π radians every 12 hours, or changes at a rate of $\frac{2\pi}{12} = \frac{\pi}{6}$ rph (radians per hour: we always work in radians and not degrees!). The minute-hand, (on the other hand?! Sorry....) has an angle that changes by 2π radians every hour, so its velocity is 2π rph. The relative velocity of the two hands is then $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rph (it is negative here since the minute hand is catching up to the hour hand at one o'clock.) This is $\frac{d\theta}{dt}$ at one o'clock, and thus we can find the rate of change of the distance s at one o'clock by plugging in this last quantity:

$$\begin{aligned}
\left.\frac{ds}{dt}\right|_{\theta=\frac{\pi}{6}} &= \left(\frac{16}{\sqrt{80-32\sqrt{3}}}\right)\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{6}} \\
&= \left(\frac{16}{\sqrt{80-32\sqrt{3}}}\right)\left(-\frac{11\pi}{6}\right) \approx -18.590\,\frac{\text{mm}}{\text{h}}.
\end{aligned}$$