EXAMPLE: PROBLEM 3.9.22 RELATED RATES

110.109 CALCULUS I (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Question 1. *A particle moves along the curve* $y = 2\sin\left(\frac{\pi x}{2}\right)$*. As the particle passes through the point* $\left(\frac{1}{3}, 1\right)$*, its* x*-coordinate increases at a rate of* $\sqrt{10}$ *cm/s. How fast is the distance from the particle to the origin changing at this moment?*

Strategy: Here, the ultimate questions asks about the distance from a point on the curve to the origin. Hence we will need the distance function: The distance r any point (x, y) is from the origin is

$$r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

If the particle is moving along the curve, then both x and y are changing while the distance is changing. But on the curve, y is related to x. Hence this is a related rates problem and we relate the distance to the x-coordinate of the point on the curve via this distance function.

Solution: We seek to relate the distance of a particle on the curve $y = 2\sin\left(\frac{\pi x}{2}\right)$ from the origin to its coordinates. The distance formula for an arbitrary point form the origin is given above in the Strategy, but if the point lies on the curve, then we can replace the y with the expression for x, and get

$$r = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2\sin\left(\frac{\pi x}{2}\right)\right)^2}.$$

To relate the rates, we simply differentiate this entire equation with respect to t. We get

$$\begin{aligned}
\frac{d}{dt}\left[r \right. &= \left. \sqrt{x^2 + \left(2\sin\left(\frac{\pi x}{2}\right)\right)^2}\right] \\
\frac{dr}{dt} &= \frac{d}{dt}\left[\sqrt{x^2 + \left(2\sin\left(\frac{\pi x}{2}\right)\right)^2}\right] \\
&= \frac{1}{2\sqrt{x^2 + \left(2\sin\left(\frac{\pi x}{2}\right)\right)^2}} \frac{d}{dt}\left[x^2 + \left(2\sin\left(\frac{\pi x}{2}\right)\right)^2\right] \\
&= \frac{1}{2\sqrt{x^2 + \left(2\sin\left(\frac{\pi x}{2}\right)\right)^2}} \left(2x + 2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\frac{\pi}{2}\right)\frac{dx}{dt}.
\end{aligned}$$

Date: November 8, 2011.

We now have everything in place. At the moment when the particle is at the point $\left(\frac{1}{3},1\right)$ (you should verify that this point is actually ON the curve), $\frac{dx}{dt}=\sqrt{10}$. Hence we get

$$\begin{aligned}
\left.\frac{dr}{dt}\right|_{\left(\frac{1}{3},1\right)} &= \left.\left(\frac{1}{2\sqrt{x^2+\left(2\sin\left(\frac{\pi x}{2}\right)\right)^2}}\left(2x+8\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\frac{\pi}{2}\right)\frac{dx}{dt}\right)\right|_{\left(\frac{1}{3},1\right)} \\
&= \frac{1}{2\sqrt{\left(\frac{1}{3}\right)^2+\left(2\sin\left(\frac{\pi\left(\frac{1}{3}\right)}{2}\right)\right)^2}}\left(2\left(\frac{1}{3}\right)+8\sin\left(\frac{\pi\left(\frac{1}{3}\right)}{2}\right)\cos\left(\frac{\pi\left(\frac{1}{3}\right)}{2}\right)\frac{\pi}{2}\right)\left(\sqrt{10}\right) \\
&= \frac{1}{2\sqrt{\frac{10}{9}}}\left(\frac{2}{3}+(4)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)\right)\sqrt{10} \\
&= 1+\frac{\sqrt{27}\pi}{2}.
\end{aligned}$$