EXAMPLE: PROBLEM 3.4.42: THE CHAIN RULE

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Question 1. *Find the derivative of the function* $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

Strategy: This derivative will be a chain rule calculation and it looks like there are multiple composite functions here. We will write this function as a composite function and use the Chain Rule to calculate the derivative piece by piece.

Solution: First, we write the function as a double composite function: Let

$$f(x) = \sqrt{x}, \quad \text{and} \quad g(x) = x + \sqrt{x + \sqrt{x}}.$$

Then

$$\frac{dy}{dx} = \frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) g'(x) = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \frac{d}{dx}\left[x + \sqrt{x + \sqrt{x}}\right].$$

This last part is a sum of functions, where the first summand is simply x and the second is another composite function of $f(x)$ again, and the inside function $h(x) = x + \sqrt{x}$. Hence we can write the function $g(x) = x + f\left(h(x)\right)$. Its derivative again uses the Chain Rule (along with the Sum Rule) to get

$$g'(x) = \frac{d}{dx}\left[x + f\left(h(x)\right)\right] = 1 + f'\left(h(x)\right) h'(x).$$

Hence we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \frac{d}{dx}\left[x + \sqrt{x + \sqrt{x}}\right] \\
&= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \frac{d}{dx}\left[x + \sqrt{x}\right]\right) \\
&= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)\right).
\end{aligned}$$

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