EXAMPLE: PROBLEM 2.5.36: LIMITS AT INFINITY

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Question 1. *Calculate* $\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1}$.

Strategy: We recognize that we cannot use the Quotient Limit Law here since the numerator does not have a limit at infinity, and the denominator has an infinite limit at infinity. However, we use the fact that the numerator never gets too large in magnitude. This allows us to say two things: (1) Since the denominator grows without bound and the numerator does not, we *guess* that the limit is 0, and (2) we can possibly use the Squeezing Theorem to see if we are correct.

Solution: First, we guess that the limit is 0. We will refer to the definition of an infinite limit to see if we are correct. Recognizing that the function is bounded, we know

$$0 \leq \sin^2 x \leq 1.$$

(Note here that the sine function lies between -1 and 1, and squared, its values lie between 0 and 1.)

Dividing by a positive quantity does not change the sense of the inequality, so

$$0 = \frac{0}{x^2+1} \leq \frac{\sin^2 x}{x^2+1} \leq \frac{1}{x^2+1}.$$

Since we already know that $\lim_{x\to\infty} 0 = 0$, we can show by the Squeezing Theorem that our guess is correct by showing that $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$.

Claim 1. $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$.

By the definition, $\lim_{x\to\infty} f(x) = L$, if for any $\epsilon > 0$, there is an $N > 0$ where,

$$\text{if } x > N, \text{ then } |f(x) - L| < \epsilon.$$

For our particular problem, given any $\epsilon > 0$, there is an N where

$$\text{if } x > N, \text{ then } \left|\frac{1}{x^2+1} - 0\right| = \frac{1}{x^2+1} < \epsilon.$$

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The idea is to use the latter ϵ-inequality to calculate the value for N. To this end, take this latter inequality and write it as an inequality with x isolated on the left:

$$\frac{1}{x^2+1} < \epsilon \quad \Rightarrow \quad x^2+1 > \frac{1}{\epsilon} \quad \Rightarrow \quad x^2 > \frac{1}{\epsilon} - 1 \quad \Rightarrow \quad x > \sqrt{\frac{1}{\epsilon} - 1}.$$

This last inequality is fine since we are looking at the limit at infinity, all input values for x will be positive on the right tail of the graph of the function. Set our value for $N = \sqrt{\frac{1}{\epsilon} - 1}$. Then, when

$$x > N = \sqrt{\frac{1}{\epsilon} - 1},$$

we will have

$$|f(x) - L| = \left| \frac{1}{x^2+1} - 0 \right| = \frac{1}{x^2+1} < \frac{1}{(N)^2+1} = \frac{1}{\left(\sqrt{\frac{1}{\epsilon}-1}\right)^2+1} = \frac{1}{\left(\frac{1}{\epsilon}-1\right)+1} = \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

Thus we have shown that the claim is true. And then by the Squeezing Theorem, since

$$\lim_{x\to\infty} 0 = 0 = \lim_{x\to\infty} \frac{1}{x^2+1}$$

we then know that $\displaystyle\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1} = 0$. Here is the graph, with the green, red, and blue curves respectively the functions $f(x) = 0$, $g(x) = \dfrac{\sin^2 x}{x^2+1}$, and $h(x) = \dfrac{1}{x^2+1}$.