EXAMPLE: PROBLEM 2.4.30 DEFINITION OF A LIMIT

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Question 1. *Show that* $\lim_{x\to 2} x^2 + 2x - 7 = 1$ *using the* ϵ-δ *definition of a limit.*

Strategy: We will employ a method similar to that of Example 2.4.4 on page 114 of the text. A good method, in general, for finding a $\delta > 0$ given an $\epsilon > 0$ is to work with the given ϵ-inequality to make it look like the δ-inequality. Then the choice of δ as a function of ϵ becomes clear.

Solution: For $f(x) = x^2 + 2x - 7$ and any choice of $\epsilon > 0$, we seek a number $\delta > 0$ where

$$\text{If } 0 < |x-2| < \delta, \quad \text{then } |f(x) - 1| = \left|(x^2 + 2x - 7) - 1\right| = \left|x^2 + 2x - 8\right| < \epsilon.$$

To calculate this δ, we manipulate the ϵ-inequality: Here

$$|f(x) - 1| = \left|x^2 + 2x - 8\right| = |x+4||x-2| < \epsilon.$$

To make this look like the δ-inequality, we need to find a way to replace the factor $|x+4|$ with a number. In essence, we need to bound it somehow (find a number surely bigger than it).

To do this, note that anywhere near the value $x = 2$, we know that the expression $x + 4$ will be near the value 6. So choose a range of input values, say $1 < x < 3$, or $|x - 2| < 1$ so that in this range, we know $3 < x + 4 < 7$. Then, as long as $|x - 2| < 1$, we can manipulate the ϵ-inequality to read

$$\left|x^2 + 2x - 8\right| = |x+4||x-2| < 7|x-2| < \epsilon$$

or $|x-2| < \frac{\epsilon}{7}$. This becomes our choice of $\delta = \frac{\epsilon}{7}$, at least when we are within 1 of the value $x = 2$. If we make this choice, then for small original choices of $\epsilon > 0$, we would have:

$$\begin{aligned} \text{If } 0 < |x-2| < \delta, \quad \text{then } |f(x) - 1| &= \left|(x^2 + 2x - 7) - 1\right| = \left|x^2 + 2x - 8\right| \\ &= |x+4||x-2| < 7|x-2| < 7\delta = 7\left(\frac{\epsilon}{7}\right) = \epsilon. \end{aligned}$$

However, this will not work when we choose $\epsilon > 0$ so large that $\frac{\epsilon}{7} > 1$, because will make our choice of $\delta = \frac{\epsilon}{7} > 1$, and we will have violated our assumption above, namely that for $|x - 2| < 1$, we would get $|x + 4| < 7$. For example, say we were given $\epsilon = 14$, so that our choice of δ was $\delta = \frac{\epsilon}{7} = 2$. Then the definition of a limit would be:

$$\begin{aligned} \text{If } 0 < |x-2| < \delta, \quad \text{then } |f(x) - 1| &= \left|x^2 + 2x - 8\right| \\ &= |x+4||x-2| < 8|x-2| < 8\delta = 16 \end{aligned}$$

which does not work when our choice of *epsilon* was 14. The reason was that for $|x - 2| < \delta = 2$, the expression $|x + 4|$ can large as 8, and not 7.

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To fix this, not that we actually have two stipulations for δ, namely $|x-2|<1$ by assumption (which a good assumption since near $x=2$ this is fine), and $|x-2|<\delta=\frac{\epsilon}{7}$ which will work for small $\epsilon>0$. Hence we can fix this completely, but making sure our choice of δ never gets large than our assumption, no matter the size of ϵ. Hence the idea that

$$\delta=\min\left\{1,\frac{\epsilon}{7}\right\}.$$

Then for large choices of $\epsilon>0$, like3 our $\epsilon=14$, which makes the calculation of $\delta=\frac{\epsilon}{7}=2$, we instead choose δ to be the smaller $\delta=1$. Then the bound we made for $|x+4|<7$ still works, and we can say,

$$\begin{aligned}\text{If } 0<|x-2|<\delta, \quad \text{then } |f(x)-1| &= \left|x^2+2x-8\right| \\ &= |x+4||x-2|<7|x-2|<7\delta=7\left(\frac{\epsilon}{7}\right)=\epsilon.\end{aligned}$$

Try this with specific numbers. You will see it works.