EXAMPLE: PROBLEM 2.4.22 DEFINITION OF A LIMIT

110.109 CALCULUS I (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Question 1. *Prove the statement, using the definition of a limit:* $\lim_{x\to -1.5} \dfrac{9-4x^2}{3+2x}$.

Strategy: We will use algebraic manipulation to simplify the quotient and then calculate a δ as a function of ϵ to satisfy the definition.

Solution: First, this rational function has s simpler form. On the domain $x \neq -1.5$, we can rewrite this function as

$$\frac{9-4x^2}{3+2x} = \frac{(3-2x)(3+2x)}{3+2x} = 3-2x \text{ on the domain } \left\{ x \in \mathbb{R} \;\middle|\; x \neq -1.5 \right\}.$$

Hence

$$\lim_{x\to -1.5} \frac{9-4x^2}{3+2x} = \lim_{x\to -1.5} 3-2x.$$

To show the limit, we must construct a $\delta > 0$, given any $\epsilon > 0$, where

$$\text{If } 0 < |x-(-1.5)| = |x+1.5| < \delta, \quad \text{then } |(3x-2)-6| < \epsilon.$$

We use the expression $|(3-2x)-6| < \epsilon$ to find δ:

$$|(3-2x)-6| = |-3-2x| = |3+2x| < \epsilon$$

implies that

$$2|x+1.5| < \epsilon \quad \text{or} \quad |x+1.5| < \frac{\epsilon}{2}.$$

This looks like our δ-inequality, which is what we want. So choose $\delta = \frac{\epsilon}{2}$. Then, given any $\epsilon > 0$, we can use this choice of δ and say:

$$\begin{aligned} \text{If } 0 < |x+1.5| < \delta, \quad \text{then } |(3x-2)-6| &= |-3-2x| = |3+2x| \\ &= 2|x+1.5| < 2\delta = 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

Date: September 26, 2011.