EXAMPLE: PROBLEM 2.3.40: LIMIT LAWS

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Question 1. *Show that* $\lim_{x\to 0^+} \sqrt{x}e^{\sin\frac{\pi}{x}} = 0.$

Strategy: We will use the Squeezing Theorem to show this since $e^{\sin\frac{\pi}{x}}$ has a problem at $x = 0$.

Solution: First, it should be clear that, for all $x \in (0, \infty)$, we have

$$-1 \leq \sin\frac{\pi}{x} \leq 1.$$

Also, since e^x is an increasing function (this means that for $x_1 > x_2$, we have $e^{x_1} > e^{x_2}$), it also follows that

$$\frac{1}{e} = e^{-1} \leq e^{\sin\frac{\pi}{x}} \leq e^1 = e \quad \text{on } (0, \infty).$$

Now, the function $\sqrt{x} > 0$ on $(0, \infty)$. Hence we do not change the "sense" of the inequality by multiplying the inequality by $\sqrt{x}$. We get

$$\frac{\sqrt{x}}{e} \leq \sqrt{x}e^{\sin\frac{\pi}{x}} \leq e\sqrt{x}.$$

Hence we have three function that satisfy this inequality, with the function we are trying to understand in the middle. The "smallest" function satisfies

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{e} = \frac{1}{e}\left(\lim_{x\to 0^+} \sqrt{x}\right) = \frac{1}{e}\sqrt{\lim_{x\to 0^+} x} = \frac{1}{e}\left(\sqrt{0}\right) = 0$$

by the Constant Multiple and Root Limit Laws. We also have

$$\lim_{x\to 0^+} e\sqrt{x} = 0$$

by the same Limit Laws. Hence by the Squeezing Theorem, we also have

$$\lim_{x\to 0^+} \sqrt{x}e^{\sin\frac{\pi}{x}} = 0.$$

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