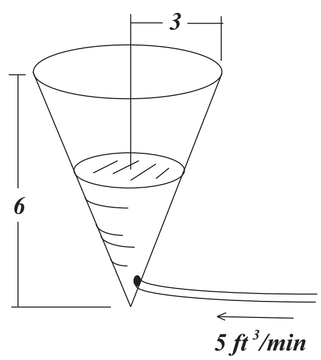
110.106 CALCULUS I: BIOLOGICAL AND SOCIAL SCIENCES

Fall 2009

A Related Rates Example: Filling a conical tank

Exercise 70 in Section 4.4 is a problem where you are to calculate a rate of increase of the height of water in a conical tank knowing only the height at the moment in question along with the constant rate of flow into the tank. Differentiating the volume equation for a right circular cone with respect to time results in a differential equation relating the rates of change of the volume and the height of water in the tank. However, there is a twist to this problem in that the volume is a function of the height of the cone AND the radius of the base of the cone (which is actually the top here; See the picture). How does one account for the fact that the resulting related rates problem will be a function also of the rate of increase in the radius of the surface of the water at any moment in time? The clue is to recognize that the radius and the height are also related. You can use this relationship between radius and height to eliminate radius from the volume equation, thus making volume a function ONLY of height. The problem becomes more straightforward then. Here is how it works:



Suppose that we pump water into an inverted right-cylindrical conical tank at the rate of 5 cubic feet per minute. The tank has a height of 6 feet and the radius at the top is 3 feet. What is the rate at which the height of the water level is rising when the water is 2 feet deep? (Note that the volume of a right cylindrical cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

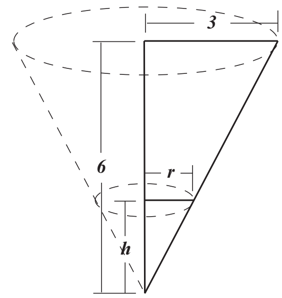
We could simply differentiate the volume equation with respect to time, obtaining a differential equation where all three of V, r, and h are changing with respect to time. Then we would get

$$\frac{dV}{dt} = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right] = \frac{1}{3}\pi \cdot \left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right).$$

However, since we at present have no way of knowing how the radius is changing with respect to time at the moment when $h = 2$, we are no closer to solving the problem. Instead, notice that there is another relationship between r and h hidden in the right cylindrical cone. Cut out a slice of the cone through the center, and one get the diagram below. Here, the depth of the water h and the radius of its surface r at any moment forms a similar triangle with the full height of the cone and the radius at the top. These two similar triangles allow us to form a ratio of their respective sides, and

$$\frac{r}{h} = \frac{3}{6}.$$

Date: October 30, 2009.



Thus we get that $r = \dfrac{1}{2}h$, and the original volume equation can be rewritten

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3.$$

Then

$$\frac{dV}{dt} = \frac{1}{12}\pi\cdot\left(3h^2\frac{dh}{dt}\right) = \frac{1}{4}\pi h^2\frac{dh}{dt}.$$

Now plug in $\dfrac{dV}{dt} = 5$ ft^3 per minute, $h = 2$ ft, and solve for $\dfrac{dh}{dt}$. You will get

$$\left.\frac{dh}{dt}\right|_{h=2} = \left.\frac{dV}{dt}\frac{4}{\pi h^2}\right|_{h=2} = 5\cdot\frac{4}{4\pi} = \frac{5}{\pi}\text{ ft/min}.$$