Homework # 9 Due W10 11/9

Section 5.2

5. $f(x) = -\frac{2}{3}x^3 + \frac{7}{2}x^2 - 3x + 4 \quad x \in \mathbb{R}$.

$f'(x) = -2x^2 + 7x - 3$

$0 = -2x^2 + 7x - 3$

$x = \frac{-7 \pm \sqrt{49 - 4(-3)(-2)}}{2(-2)} = \frac{-7 \pm \sqrt{49-24}}{-4} = \frac{-7 \pm \sqrt{25}}{-4} = \frac{-7 \pm 5}{-4}$

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So, $x = \frac{-7+5}{-4} = \frac{-2}{-4} = 1/2$ and $x = \frac{-7-5}{-4} = \frac{-12}{-4} = 3$

Apply first derivative test:

$(-) \quad \underset{1/2}{|} \quad (+) \quad \underset{3}{|} \quad (-)$

$f'(0) = -3 < 0$

$f'(1) = -2+7-3 = 2 > 0$

$f'(4) = -32 + 28 - 3 = -7 < 0$

Thus, $f(x)$ is increasing on $(1/2, 3)$

and decreasing on $(-\infty, 1/2) \cup (3, \infty)$

$f''(x) = -4x + 7$

$0 = -4x + 7$

$4x = 7$

$x = 7/4$

Applying second derivative test:

$(+) \quad \underset{7/4}{|} \quad (-)$

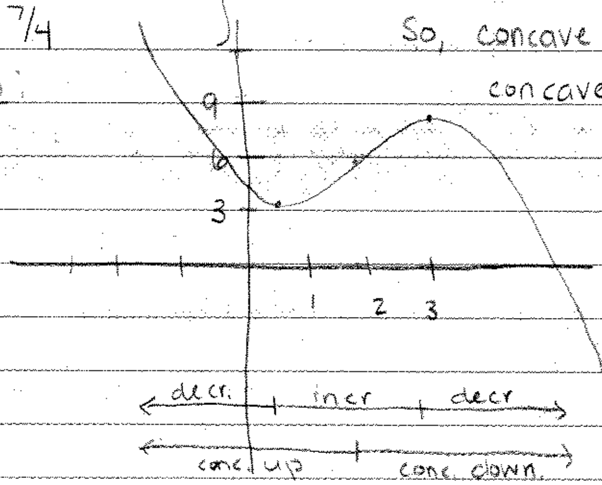
$f''(0) = 7 > 0$

$f''(2) = -1 < 0$

So, concave up on $(-\infty, 7/4)$

concave down on $(7/4, \infty)$

Graph:



8. $f(x) = (2x-3)^{1/3} \quad x \in \mathbb{R}$

$f'(x) = \frac{1}{3}(2x-3)^{-2/3}(2) = \frac{2}{3}(2x-3)^{-2/3}$

$0 = \frac{2}{3}(2x-3)^{-2/3}$ has no solution.

Other critical points occur when $2x-3=0 \Rightarrow x = 3/2$.

Applying the first derivative test:

$\underset{3/2}{(+) \quad | \quad (+)}$ $\quad f'(0) > 0 \quad\quad x \neq 3/2$

$f'(2) > 0 \quad$ $f(x)$ is always increasing

$f''(x) = -2/3(2/3)(2x-3)^{-5/3}(2) = -8/9(2x-3)^{-5/3}$

$0 = -8/9(2x-3)^{-5/3}$ has no solution.

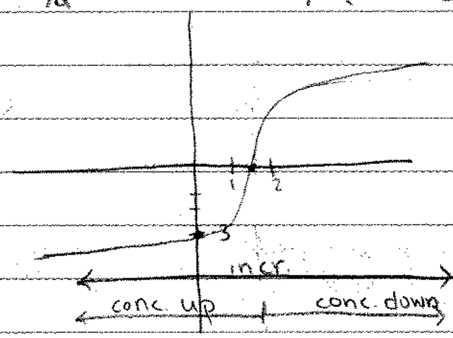
Other critical points occur when $2x-3=0 \Rightarrow x = 3/2$.

Applying the second derivative test:

$\underset{3/2}{(+) \quad | \quad (-)}$ $\quad f''(0) > 0 \quad$ $f(x)$ is concave up on $(-\infty, 3/2)$

$f''(2) < 0 \quad$ $f(x)$ is concave down on $(3/2, \infty)$.

Graph:



12. $f(x) = y = \frac{1}{1+x} \quad x \neq -1$

$f'(x) = -(1+x)^{-2}$

$0 = -\frac{1}{(1+x)^2}$ has no sol'n

Other critical point at $x \neq -1$.

Using first derivative test: $f'(-2) < 0$

$\underset{-1}{(-) \quad | \quad (-)}$ $\quad f'(0) < 0$

$f(x)$ is always decreasing $x \neq -1$

$f''(x) = 2(1+x)^{-3}$

$0 = \frac{2}{(1+x)^3}$ has no sol'n, use $x=-1$.

By second derivative test:

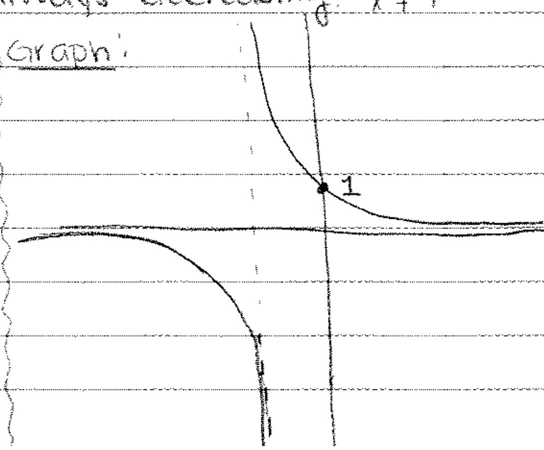
$\underset{-1}{(-) \quad | \quad (+)}$

$f''(-2) < 0 \; ; \; f''(0) > 0$

$f(x)$ is concave up on $(-1, \infty)$

$f(x)$ is concave down on $(-\infty, -1)$

Graph:



14. $f(x) = y = \frac{x}{x+1} = x(x+1)^{-1} \quad x \geq 0$

$f'(x) = (x+1)^{-1} + (-(x+1)^{-2})x = \frac{1}{x+1} - \frac{x}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

$0 = \frac{1}{(x+1)^2}$ has no solution, but $f'(x)$ is undefined at $x = -1$, since this is not in our domain, we disregard

Applying first derivative test:

$f'(1) = \frac{1}{2^2} > 0$ So, $f(x)$ is increasing on $[0, \infty)$

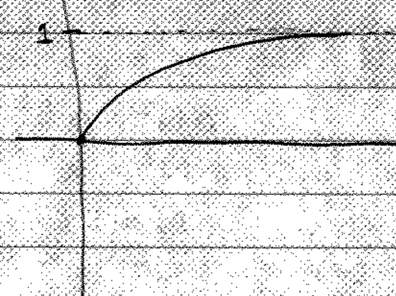
$f''(x) = -2(x+1)^{-3} = -\frac{2}{(x+1)^3}$

$0 = -\frac{2}{(x+1)^3}$ has no solution, and again $x = -1$ is not in the domain.

Applying the second derivative test, $f''(1) = -\frac{2}{2^3} < 0$

So, $f(x)$ is always concave down.

Graph:



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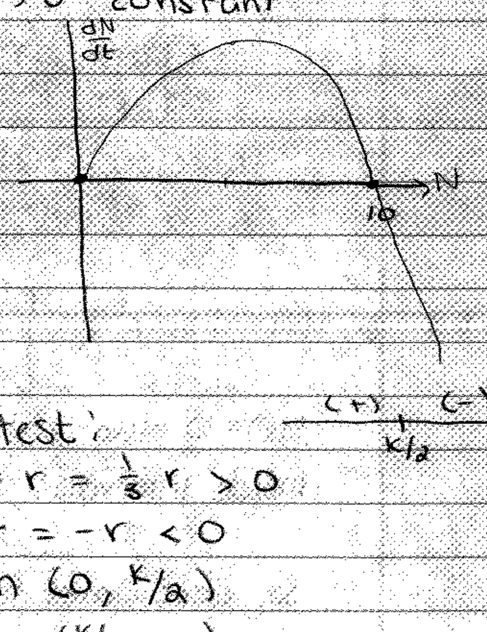
26. $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \quad t \geq 0, \qquad r, K > 0$ constant

(a) $\frac{dN}{dt} = 3N\left(1 - N/10\right) \longrightarrow$



(b) $f(N) = rN\left(1 - N/K\right)$

$= rN - \frac{r}{K}N^2$

$f'(N) = r - \frac{2r}{K}N$

$0 = r - \frac{2r}{K}N$

$\frac{2r}{K}N = r$

$\frac{2}{K}N = 1 \Rightarrow N = \frac{K}{2}$

$\underset{K/2}{\overset{(+) \quad | \quad (-)}{\overline{\qquad\qquad}}}$

Applying the first derivative test:

$f'(K/3) = r - \frac{2r}{K}\left(\frac{K}{3}\right) = r - \frac{2}{3}r = \frac{1}{3}r > 0$

$f'(K) = r - \frac{2r}{K}(K) = r - 2r = -r < 0$

So, $f(N)$ is increasing on $(0, K/2)$ and decreasing on $(K/2, \infty)$.

28. $f(R) = \frac{aR}{k+R} \quad R \geq 0 \quad a, k > 0$ constant

$f'(R) = \frac{a(k+R) - aR(1)}{(k+R)^2} = \frac{ak + aR - aR}{(k+R)^2} = \frac{ak}{(k+R)^2}$

$0 = \frac{ak}{(k+R)^2}$ has no solution. So, since there are no other critical points in the domain, we see using the first derivative test that

$f'(1) = \frac{ak}{(k+1)^2} > 0 \Rightarrow f(R)$ is always increasing.

And thus, never decreasing

30. $f(N) = \frac{aN}{k^2+N^2} \quad a, k > 0$ constant

$f'(N) = \frac{a(k^2+N^2) - aN(2N)}{(k^2+N^2)^2} = \frac{ak^2 + aN^2 - 2aN^2}{(k^2+N^2)^2} = \frac{ak^2 - aN^2}{(k^2+N^2)^2}.$

$0 = \frac{ak^2 - aN^2}{(k^2+N^2)^2} \Rightarrow 0 = ak^2 - aN^2$

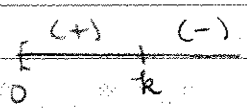
$aN^2 = ak^2$

$N^2 = k^2$

$N = \pm k$

(reject $-k$ as $N>0$)

Since there are no other critical points, we proceed with the first deriv. test:



$f'(k/2) = \frac{ak^2 - a(k/2)^2}{(k^2+(k/2)^2)^2} = \frac{ak^2 - \frac{1}{4}ak^2}{(k^2+\frac{1}{4}k^2)^2} = \frac{3/4\, ak^2}{(5/4\, k^2)^2} > 0$

$f'(2k) = \frac{ak^2 - a(2k)^2}{(k^2+(2k)^2)^2} = \frac{ak^2 - 4ak^2}{(5k^2)^2} = \frac{-3ak^2}{25k^4} < 0$

Therefore, $f(N)$ is increasing on $(0, k)$ and decreasing on (k, ∞)

Section 5.3

6. $f(x) = |16 - x^2| = \begin{cases} 16 - x^2 & x \in [-4, 4] \\ x^2 - 16 & x \in [-5, -4) \cup (4, 8] \end{cases}$

$f(x) = 0 = |16 - x^2| \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Inside Intervals: $x \in (-4, 4) \Rightarrow f'(x) = -2x \Rightarrow 0 = -2x \Rightarrow x = 0.$

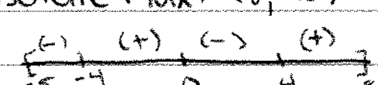
$x \in [-5, 4) \cup (4, 8] \Rightarrow 2x = f'(x) \Rightarrow 0 = 2x \Rightarrow x = 0 \notin [-5,-4) \cup (4,8]$

So, candidates for extrema are: $x = \pm 4, 0, -5, 8.$

Local Min: $x = \pm 4$ — Absolute Min: $(4,0), (-4,0)$

Local Max: $x = 0, -5, 8$ — Absolute Max: $(8, 48)$

Using first derivative test:



$f(-4.5) < 0, \; f(-1) > 0, \; f(1) < 0, \; f(5) > 0$

So, $f(x)$ is decreasing on $[-5, -4) \cup (0, 4)$; increasing on $(-4, 0) \cup (4, 8]$.

* 10. $f(x) = y = \ln(x^2+1) - x$, $x \in \mathbb{R}$.

$f'(x) = \frac{2x}{x^2+1} - 1$

$0 = \frac{2x}{x^2+1} - 1$

$1 = \frac{2x}{x^2+1}$

$x^2+1 = 2x$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$\underline{x = 1}$

$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$

$= \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$

$f''(1) = \frac{2-2}{(2)^2} = 0$

Therefore, there are no maxima or minima, either local or global.

$f'(0) = \frac{0}{0+1} - 1 = -1 < 0 \Rightarrow f(x)$ is always decreasing

14. $f(x) = y = x^2(1-x) = x^2 - x^3$ $\quad x \in \mathbb{R}$

$f'(x) = 2x - 3x^2$

$0 = 2x - 3x^2$

$0 = x(2-3x)$

$x = 0 \quad x = 2/3$

$f''(x) = 2 - 6x$

By second derivative test for extrema:

$f''(0) = 2 > 0 \Rightarrow x = 0$ is a local min.

$f''(2/3) = 2 - 6(2/3) = 2 - 2(2) = 2 - 4 = -2 < 0$

$\Rightarrow x = 2/3$ is a local max.

Since there are no other candidates for extrema, $(0,0)$ is the absolute min and $(2/3, 4/27)$ is the absolute max

Therefore, f is increasing on $(0, 2/3)$ and decreasing on $(-\infty, 0) \cup (2/3, \infty)$

GRADE 24. $f(x) = \ln x + \frac{1}{x}$, $x > 0$

$f'(x) = \frac{1}{x} + (-x^{-2}) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$

$f''(x) = \frac{x^2 - 2x(x-1)}{x^4} = \frac{x^2 - 2x^2 + 2x}{x^4} = \frac{2x - x^2}{x^4} = \frac{2-x}{x^3}$

$0 = \frac{2-x}{x^3} \iff 0 = 2 - x \iff x = 2.$

Use 2^{nd} derivative test: $\quad \underset{2}{\overset{(+) \quad (-)}{\longleftarrow\!\!\!|\!\!\!\longrightarrow}}$

$f''(1) = {}^{2-1}/_{1^3} = {}^{1}/_{1} = 1 > 0$

$f''(3) = {}^{2-3}/_{3^3} = -{}^{1}/_{27} < 0$

$\therefore x = 0$ is an inflection point.

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26. $N(t) = \frac{100}{1+3e^{-2t}} \quad t \geq 0$

(a) $N(0) = \frac{100}{1+3e^{-2(0)}} = \frac{100}{1+3(1)} = \frac{100}{1+3} = \frac{100}{4} = 25.$

(b) $N'(t) = \frac{-100(-6e^{-2t})}{(1+3e^{-2t})^2} = 600e^{-2t}/(1+3e^{-2t})^2$

$0 = 600e^{-2t}/(1+3e^{-2t})^2$ has no solution and is always defined since $e^{-2t} > 0$.

This also shows that $N'(t) > 0$ and thus is always increasing.

(c) $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{100}{1+3e^{-2t}} = 100\left(1/\lim_{t\to\infty}(1+3e^{-2t})\right)$

$= 100/(\lim_{t\to\infty} 1 + 3\lim_{t\to\infty} e^{-2t}) = 100/1+3(0) = 100/1+0 = 100.$

(d) Notice that $N'(t) = \frac{6e^{-2t}}{1+3e^{-2t}} N(t)$

Then, $N''(t) = N'(t)\left(\frac{6e^{-2t}}{1+3e^{-2t}}\right) + N(t)\left(\frac{-12e^{-2t}(1+3e^{-2t}) - (-6e^{-2t})(6e^{-2t})}{(1+3e^{-2t})^2}\right)$

$N''(t) = \frac{36e^{-4t}}{(1+3e^{-2t})^2} N(t) + \left(\frac{-12e^{-2t} - 36e^{-4t} + 36e^{-4t}}{(1+3e^{-2t})^2}\right) Nt$

$N''(t) = \left(\frac{36e^{-4t} - 12e^{-2t}}{(1+3e^{-2t})^2}\right) N(t)$

$0 = \frac{36e^{-4t} - 12e^{-2t}}{(1+3e^{-2t})^2} N(t)$

$0 = 36e^{-4t} - 12e^{-2t}$ or $\cancel{0 = N(t)}$

$0 = 3e^{-4t} - e^{-2t} = e^{-4t}(3 - e^{2t})$

$\cancel{0 = e^{-4t}}$ or $0 = 3 - e^{2t}$

$0 = 3 - e^{2t}$

$3 = e^{2t}$

$\ln 3 = 2t$

$\frac{\ln 3}{2} = t$

$N(t) = \frac{100}{1+3e^{-2t}}$

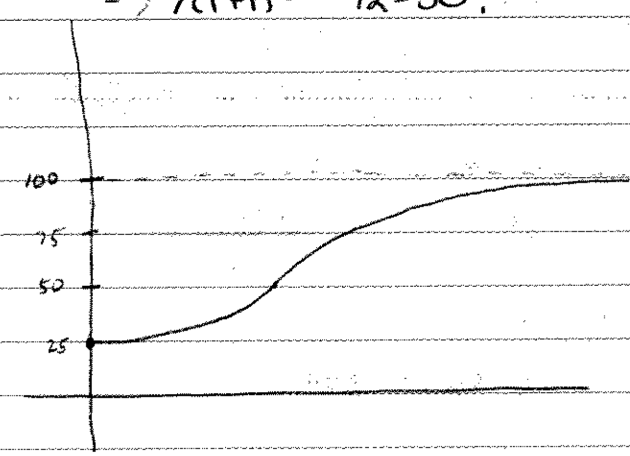
$N\left(\frac{\ln 3}{2}\right) = 100/(1+3e^{-2(\ln 3)/2})$

$= 100/(1+3e^{\ln 3^{-1}})$

$= 100/(1+3(1/3))$

$= 100/(1+1) = 100/2 = 50.$

(e)



28. $f(x) = y = x^4 - 2x^2 \qquad x \in \mathbb{R}$

$f'(x) = 4x^3 - 4x$

$0 = 4x(x^2-1) = 4x(x+1)(x-1)$

$4x = 0 \qquad x+1=0 \qquad x-1=0$

$x=0 \qquad x=-1 \qquad x=1$

There are no other candidates for extrema.

$f''(x) = 12x^2 - 4$

$0 = 12x^2 - 4$

$0 = 3x^2 - 1$

$1/3 = x^2$

$x = \pm 1/\sqrt{3} = \pm \sqrt{3}/3$. ← Inflection Points

Apply second derivative test for extrema.

$f''(0) = -4 < 0 \Rightarrow x=0$ is a local max.

$f''(-1) = 8 > 0 \Rightarrow x=-1$ is a local min

$f''(1) = 8 > 0 \Rightarrow x=1$ is a local min.

$f(0)=0 \Rightarrow (0,0)$ is the absolute max.

$\left.\begin{array}{l} f(-1)=-1 \\ f(1)=-1 \end{array}\right\} \Rightarrow \begin{array}{l}(-1,-1) \\ (1,-1)\end{array}$ are both absolute mins

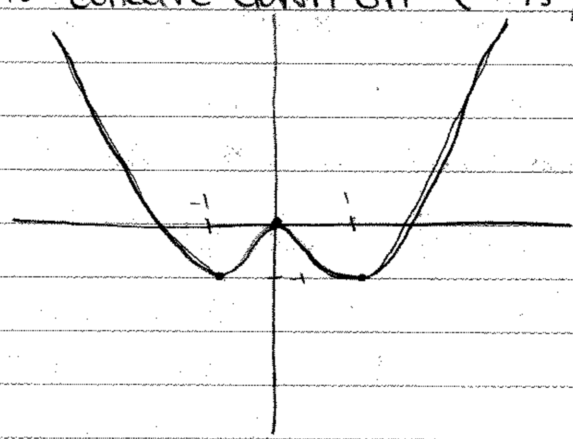
Based on Above:

f is increasing on $(-1,0) \cup (1,\infty)$

f is decreasing on $(-\infty,-1) \cup (0,1)$

f is concave up on $(-\infty, -\sqrt{3}/3) \cup (\sqrt{3}/3, \infty)$

f is concave down on $(-\sqrt{3}/3, \sqrt{3}/3)$



36. $f(x) = -\frac{2}{x^2-1} \quad x \neq -1, 1$

(a) $\lim_{x\to +\infty} f(x) = 0$ and $\lim_{x\to -\infty} f(x) = 0$

by a degree argument.

(b) $\lim_{x\to -1^-} f(x) = \left(\lim_{x\to -1^-} \left(\frac{1}{x+1}\right)\right)\left(\lim_{x\to -1^-} \left(\frac{-2}{x-1}\right)\right)$

$= \left(\lim_{x\to -1^-} \left(\frac{1}{x+1}\right)\right)\left(\frac{-2}{-1-1}\right) = \lim_{x\to -1^-} \frac{1}{x+1} = -\infty.$

$\lim_{x\to -1^+} f(x) = \left(\lim_{x\to -1^+} \frac{1}{x+1}\right)\left(\lim_{x\to -1^+} \frac{-2}{x-1}\right)$

$= \left(\lim_{x\to -1^+} \frac{1}{x+1}\right)\left(\frac{-2}{-1-1}\right) = \left(\lim_{x\to -1^+} \frac{1}{x+1}\right) = \infty$

$\lim_{x\to 1^-} f(x) = \left(\lim_{x\to 1^-} \frac{1}{x-1}\right)\left(\lim_{x\to 1^-} \frac{-2}{x+1}\right)$

$= \left(\lim_{x\to 1^-} \frac{1}{x-1}\right)\left(\frac{-2}{1+1}\right) = -\lim_{x\to 1^-} \frac{1}{x-1} = \infty$

$\lim_{x\to 1^+} f(x) = \left(\lim_{x\to 1^+} \frac{1}{x-1}\right)\left(\lim_{x\to 1^+} \frac{-2}{x+1}\right)$

$= \left(\lim_{x\to 1^+} \frac{1}{x-1}\right)\left(\frac{-2}{1+1}\right) = -\lim_{x\to 1^+} \frac{1}{x-1} = -\infty.$

only?? (c) $f'(x) = -2\left(-(x^2-1)^{-2}(2x)\right) = 4x/(x^2-1)^2$

$f'(x) = 0 = 4x/(x^2-1)^2$

$0 = 4x$

$x = 0$

~~Apply first derivative test:~~

$(-) \quad (-) \quad (+) \quad (+)$

$-1 \quad 0 \quad 1$

$f'(-2) = {-8}/{(4-1)^2} < 0$

$f'(-\frac{1}{2}) = {-2}/{(\frac{1}{4}-1)^2} < 0$

$f'(\frac{1}{2}) > 0$

$f'(2) > 0$

f is increasing on $(0,1)\cup(1,\infty)$

and decreasing on $(-\infty,-1)\cup(-1,0)$

$x=0$ is a local min $(0,2)$

(d) $f''(x) = \frac{4(x^2-1)^2 - 4x(2(x^2-1)(2x))}{(x^2-1)^{4\,3}} = \frac{4x^2-4-16x^2}{(x^2-1)^3} = -\frac{12x^2+4}{(x^2-1)^3}$

$0 = f''(x) = -\frac{12x^2+4}{(x^2-1)^3}$ has no solutions.

Thus, f has no inflection points.

Using the second derivative test,

$(-) \quad (+) \quad (-)$

$-1 \quad 1$

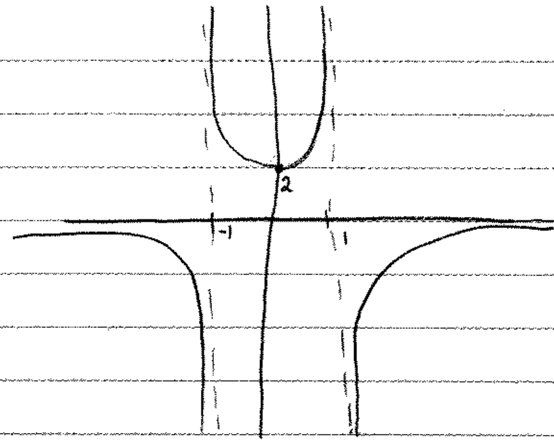
$f''(-2) < 0$

$f''(0) > 0$

$f''(2) < 0$

Thus, $f(x)$ is concave up on $(-1,1)$ and concave down on $(-\infty,-1)\cup(1,\infty)$.

(c)



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44. $f(N) = aN/(k^2+N^2)$

By Exercise 5.2.30 above, $f'(N) = \dfrac{ak^2 - aN^2}{(k^2+N^2)^2}$

and $f'(N) = 0$ when $N = k$.

Then $f''(N) = \dfrac{(-2aN)(k^2+N^2)^{\cancel{2}} - (ak^2 - aN^2)(2\cancel{(k^2+N^2)}(2N))}{(k^2+N^2)^{\cancel{4}3}}$

$= \dfrac{-2ak^2N - 2aN^3 - 4ak^2N + 4aN^3}{(k^2+N^2)^3}$

$= \dfrac{2aN^3 - 6ak^2N}{(k^2+N^2)^3}$

So, using the second derivative test for extrema, we see that:

$f''(k) = \dfrac{2ak^3 - 6ak^3}{(2k^2)^3} = -4ak^3/8k^6 < 0$

Thus, $\underline{N=k \text{ is a local maximum}}$.