Homework #10 Due W/D 11/16

Section 5.4

2. Rectangle


$P = 2x + 2y \Rightarrow y = \frac{1}{2}P - x$

$A = xy = x\left(\frac{1}{2}P - x\right) = \frac{1}{2}Px - x^2$

Note: P is a constant

maximize Area: $A(x) = \frac{1}{2}Px - x^2$

$A'(x) = \frac{1}{2}P - 2x \longrightarrow A''(x) = -2 < 0$ ∴ By second derivative test for extrema, $A''(\frac{1}{4}P) = -2 < 0$, so it is a maximum.

$0 = \frac{1}{2}P - 2x$

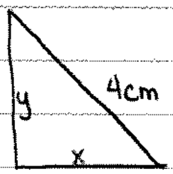
$2x = \frac{1}{2}P$

$x = \frac{1}{4}P.$

So, if $x = \frac{1}{4}P$, Then $y = \frac{1}{2}P - \frac{1}{4}P = \frac{1}{4}P.$

Therefore, since $x = y$, the maximum area occurs in the case of $\boxed{\text{a square.}}$

6.




Pythagorean's Theorem: $x^2 + y^2 = 4^2$

So, $y = \sqrt{16 - x^2}$

$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{16 - x^2}.$

Choose $x, y > 0$ (Dimensions can't be negative.

Domain: $0 < x < 4$, $0 < y < 4$.

maximize Area: $A(x) = \frac{1}{2}x\sqrt{16 - x^2}$

$A'(x) = \frac{1}{2}\sqrt{16 - x^2} + \frac{1}{2}x\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right)$

$= \frac{1}{2}\sqrt{16 - x^2} - \frac{1}{2}x^2(16 - x^2)^{-1/2}$

$= \frac{\sqrt{16 - x^2}}{2} - \frac{x^2}{2\sqrt{16 - x^2}} = \frac{16 - x^2 - x^2}{2\sqrt{16 - x^2}} = \frac{16 - 2x^2}{2\sqrt{16 - x^2}} = \frac{8 - x^2}{\sqrt{16 - x^2}}$

$0 = \frac{8 - x^2}{\sqrt{16 - x^2}} \iff 0 = 8 - x^2 \iff x^2 = 8 \iff x = \pm 2\sqrt{2}.$

Since $x > 0$, so $x = 2\sqrt{2}$ is in the domain.

By the FDT,

$\underset{2\sqrt{2} = \sqrt{8}}{\overset{(+) \qquad (-)}{\longleftarrow\!\!+\!\!\longrightarrow}}$

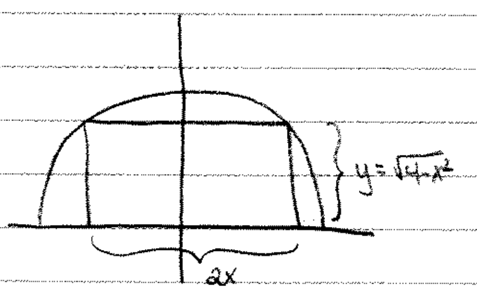
$A'(1) = \frac{8 - 1^2}{\sqrt{16 - 1^2}} = \frac{7}{\sqrt{15}} > 0$

$A'(3) = \frac{8 - 3^2}{\sqrt{16 - 3^2}} = \frac{-1}{\sqrt{7}} < 0$ ∴ $x = 2\sqrt{2}$ is a maximum.

So, the largest possible area is:

$A(2\sqrt{2}) = \frac{1}{2}(2\sqrt{2})\sqrt{16 - (2\sqrt{2})^2} = \sqrt{2}\sqrt{16 - 4 \cdot 2} = \sqrt{2}\sqrt{16 - 8} = \sqrt{2}\sqrt{8} = \sqrt{16} = \boxed{4\,cm^2}$

10.




$0 < x < 2$

$0 < y < 2$

$A = 2xy = 2x(\sqrt{4 - x^2})$

$A'(x) = 2\sqrt{4 - x^2} + 2x\left(\frac{1}{2}(4 - x^2)^{-1/2}(-2x)\right)$

$= \frac{2(4 - x^2)}{\sqrt{4 - x^2}} - \frac{2x^2}{\sqrt{4 - x^2}} = \frac{8 - 2x^2 - 2x^2}{\sqrt{4 - x^2}} = \frac{8 - 4x^2}{\sqrt{4 - x^2}}.$

$0 = \frac{8 - 4x^2}{\sqrt{4 - x^2}} \iff 0 = 8 - 4x^2 \iff x^2 = 2 \iff x = \pm\sqrt{2}$

Only $x = \sqrt{2}$ is in domain. Use FDT: $\underset{\sqrt{2}}{\overset{(+) \qquad (-)}{\longleftarrow\!\!+\!\!\longrightarrow}}$

$A'(1) = \frac{8 - 4}{\sqrt{3}} = \frac{4}{\sqrt{3}} > 0$ ; $A'(\sqrt{3}) = \frac{8 - 4(3)}{\sqrt{4 - 3}} = \frac{-4}{1} < 0$

Therefore, $x = \sqrt{2}$ is a maximum and

$A(\sqrt{2}) = 2(\sqrt{2})(\sqrt{4 - 2}) = 2(\sqrt{2})(\sqrt{2}) = 2(2) = \boxed{4.}$

18. Smallest $\Rightarrow$ minimum

Given Area A $\Rightarrow$ A is constant.

$A = \frac{1}{2}r^2\theta$ Solve for $\theta$.

$2A = r^2\theta$

$\theta = 2A/r^2$ plug in

Perimeter $= P = 2r + s = 2r + r\theta$

$P(r) = 2r + 2A/r^2 \cdot r = 2r + 2A/r$

$P'(r) = 2 + 2A(-1/r^2) = 2 - 2A/r^2$

$0 = 2 - 2A/r^2$

$2A/r^2 = 2$

$2A = 2r^2$

$A = r^2 \Rightarrow r = \pm\sqrt{A}$

Since $A \geq 0$, $r = \sqrt{A}$.

Apply SDT (Extrema): $P''(r) = -2A(-2/r^3) = 4A/r^3$. Then, $P''(\sqrt{A}) = 4A/(\sqrt{A})^3 = 4A/A\sqrt{A} = 4/\sqrt{A} > 0$. Thus, $r = \sqrt{A}$ is a minimum.

Solve for $\theta$ when $r = \sqrt{A}$: $\theta = 2A/(\sqrt{A})^2 = 2A/A = 2$

So $\boxed{r = \sqrt{A},\ \theta = 2}$

Section 5.5

5. $\lim_{x\to 0} \frac{\sqrt{2x+4}-2}{x}$ Indeterminate Form: $0/0$

So, $\lim_{x\to 0} \frac{\sqrt{2x+4}-2}{x} \underset{\text{L'hospital}}{=} \lim_{x\to 0} \left(\frac{1}{2}(2x+4)^{-1/2}(2)\right)/(1) = \lim_{x\to 0} (2x+4)^{-1/2} = 4^{-1/2} = \boxed{1/2}$

14. $\lim_{x\to 0} \frac{2^{-x}-1}{5^x-1}$ Indeterminate Form: $0/0$

So, $\lim_{x\to 0} \frac{2^{-x}-1}{5^x-1} \underset{\text{L'hospital}}{=} \lim_{x\to 0} 2^{-x}(-1)(\ln 2)/5^x(\ln 5) = \boxed{\frac{-\ln 2}{\ln 5}}$

15. $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$ Indeterminate Form: $0/0$

So, $\lim_{x\to 0} \frac{e^x-1-x}{x^2} \underset{\text{L'hospital \#1}}{=} \lim_{x\to 0} \frac{e^x-1}{2x}$ Indeterminate Form: $0/0$

So, $\lim_{x\to 0} \frac{e^x-1}{2x} \underset{\text{L'hospital \#2}}{=} \lim_{x\to 0} \frac{e^x}{2} = \boxed{1/2}$ *Double Use

30. $\lim_{x\to\infty} (x - \sqrt{x^2-1})$ Indeterminant Form: $\infty - \infty$

So, $\lim_{x\to\infty} (x - \sqrt{x^2-1}) = \lim_{x\to\infty} x\left(1 - \sqrt{1 - 1/x^2}\right) = \lim_{x\to\infty} \frac{1-\sqrt{1-1/x^2}}{1/x}$ Indeterminate Form: $0/0$

So, $\lim_{x\to\infty} \frac{1-\sqrt{1-1/x^2}}{1/x} = \lim_{x\to\infty} \frac{-\frac{1}{2}(1-x^{-2})^{-1/2}(-2x^{-3})}{-1/x^2}$

$= \lim_{x\to\infty} \frac{-x^2}{x^3(1-x^{-2})^{1/2}} = \lim_{x\to\infty} \frac{-1}{x(1-x^{-2})^{1/2}} = \boxed{0}$

34. $\lim_{x\to0^+} x^{\sin x}$ Indeterminate Form: $0^0$

$\lim_{x\to0^+} x^{\sin x} = \lim_{x\to0^+} \exp(\ln x^{\sin x}) = \lim_{x\to0^+} \exp(\sin x \ln x)$

$= \exp(\lim_{x\to0^+} \sin x \ln x)$

since $e^x$ is continuous

$\lim_{x\to0^+} \sin x \ln x$ Indeterminate Form: $0\cdot\infty$

So, $\lim_{x\to0^+} \sin x \ln x = \lim_{x\to0^+} \ln x / \tfrac{1}{\sin x} = \lim_{x\to0^+} \ln x / \csc x$ Indeterminate Form: $\infty/\infty$.

$= \lim_{x\to0^+} \tfrac{1}{x} / -\csc x \cot x = \lim_{x\to0^+} -\frac{\sin x \tan x}{x}$ Indeterminate Form: $0/0$.

$= \lim_{x\to0^+} -\frac{\cos x \tan x + \sin x \sec^2 x}{1}$

$= \lim_{x\to0^+} -\sin x - \sin x \sec^2 x = 0.$

So, $\lim_{x\to0^+} x^{\sin x} = \exp(\lim_{x\to0^+} \sin x \ln x) = \exp(0) = \boxed{1.}$

48. $\lim_{x\to0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = \lim_{x\to0^+} \frac{1}{x}(1-\sqrt{x})$

$= \lim_{x\to0^+} \frac{1-\sqrt{x}}{x} \longrightarrow \boxed{\infty \text{ (or DNE).}}$

52. $\lim_{x\to\infty} (1+\frac{c}{x})^x = \lim_{x\to\infty} \exp(\ln(1+\frac{c}{x})^x) = \lim_{x\to\infty} \exp(x\ln(1+c/x))$

$= \exp(\lim_{x\to\infty} x\ln(1+c/x))$

$\lim_{x\to\infty} x\ln(1+c/x) = \lim_{x\to\infty} \frac{\ln(1+c/x)}{1/x}$ Indeterminate Form: $0/0$.

$= \lim_{x\to\infty} \left(\frac{-c/x^2}{1+c/x}\right) / (-1/x^2)$ Apply L'hospital.

$= \lim_{x\to\infty} \left(c / 1+c/x\right) = c$

So, $\lim_{x\to\infty} (1+\frac{c}{x})^x = \exp(\lim_{x\to\infty} x\ln(1+c/x)) = \exp(c) \boxed{= e^c}$

Section 5.8

8. $f(x) = x - 2x^2 - 3x^3 - 4x^4$

$F(x) = \frac{1}{2}x^2 - \frac{2}{3}x^3 - \frac{3}{4}x^4 - \frac{4}{5}x^5 + C$

12. $f(x) = x^3 - 1/x^3 = x^3 - x^{-3}$

$F(x) = \frac{1}{4}x^4 - \frac{1}{-2}x^{-2} + C = \frac{1}{4}x^4 + \frac{1}{2}x^{-2} + C$

14. $f(x) = \frac{x}{1+x} = \frac{1+x-1}{1+x} = \frac{1+x}{1+x} - \frac{1}{1+x} = 1 - \frac{1}{1+x}$

$F(x) = x - \ln(1+x) + C$

22. $f(x) = -3e^{-4x}$

$F(x) = \frac{3}{4}e^{-4x} + C$

32. $f(x) = \sec^2(-4x)$

$F(x) = -\frac{1}{4}\tan(-4x) + C$

42. $\frac{dy}{dt} = t^2(1-t^2) \quad t \geq 0$

$= t^2 - t^4$

$y = \frac{1}{3}t^3 - \frac{1}{5}t^5 + C$

62. $\frac{dN}{dt} = t^{-1/3} \quad t > 0 \quad N(0) = 60.$

$N = \frac{3}{2}t^{2/3} + C$

$N(0) = \frac{3}{2}(0)^{2/3} + C$

$60 = C$ $\quad \therefore N(t) = \frac{3}{2}t^{2/3} + 60.$

65. $h_0 = 0$, $h_f = 100\text{ft}$ dropped $\Rightarrow v_0 = 0.$

$a = 32 \text{ft/s}^2$

when will the object hit the ground, and at what speed $\Rightarrow$ we need a position function and velocity function:

$\frac{dv}{dt} = a(t) = 32 \Rightarrow v(t) = 32t + v_0 = 32t$

$\frac{ds}{dt} = v(t) = 32t \Rightarrow s(t) = 16t^2 + h_0 = 16t^2$

So, find $t$ when $s(t) = 100$. $100 = 16t^2 \Rightarrow \frac{100}{16} = t^2 \Rightarrow \pm\frac{10}{4} = t$.

$t \geq 0$, so, $t = 10/4 = \boxed{5/2 \text{ seconds}}$

Plug $t = 5/2$ s into velocity: $v(5/2) = 32(5/2) = 16(5) = \boxed{80 \text{ ft/s}}$

67. (a) $\frac{dV}{dt} = -at(24-t) + 4$ since the first term represents the loss in water (proportional to $t(24-t)$) and the second term is the amount of water being added (at a constant rate).

(b) $\frac{dV}{dt} = -24at + at^2 + 4$

$V = -12at^2 + \frac{1}{3}at^3 + 4t$

$V(24) = (24)(-12a(24) + \frac{1}{3}(24)^2 + 4) = 0$

$-12a(24) + \frac{a}{3}(24)^2 = -4$

$-288a + \frac{1}{3}a(576) = -4$

$-288a + 192a = -4$

$96a = 4$

$a = 4/96$