110.106 CALCULUS I: BIOLOGICAL AND SOCIAL SCIENCES

Fall 2009
A Leibniz Rule Example with a domain problem

You may have noticed that one of the problems in the recent homework problem set has a twist to it. If you did, then how you handle the twist can be quite important. I thought I would elaborate on **Problem 6.2.32** here. Note that we will not be grading this problem. However, it is a good problem to generate a bit of deeper thinking and some good discussion.

Exercise 6.2.32) Use the Leibniz Rule to find $\frac{dy}{dx}$ for $y = \int_{1+x^2}^{2} \tan t \; dt$.

Solution: This problem should be a straightforward application of the Leibniz Rule, which states that, for $g(x)$ and $h(x)$ differentiable functions, and $f(t)$ continuous for t between the limits $g(x)$ and $h(x)$, then $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t) \; dt = f\left[h(x)\right]h'(x) - f\left[g(x)\right]g'(x)$. The only problem here is that for certain choices of x, the function $f(t) = \tan t$ is not continuous. This happens whenever a value for x is chosen so that there is a vertical asymptote in between $g(x) = 1 + x^2$ and $h(x) = 2$. For example, choosing $x = 0$ doesn't work because the vertical asymptote of $\tan t$ is at $t = \frac{\pi}{2} \cong 1.67$, which lies between $1 + q^2 = 1$ and 2. But this ONLY means that the antiderivative $F(x) = \int_{1+x^2}^{2} \tan t \; dt$ is not defined at $x = 0$ (that is, $x = 0$ is not in the domain of $F(x)$). This is sort of like saying that the function $\sqrt{x-1}$ is not defined at $x = 0$.

Indeed, the integrand is fine for all values of x where the limits do not lie on either side of one of the asymptotes of $\tan t$. Since the upper limit is 2, the lower limit will have to solve the equation

$$\frac{\pi}{2} < 1 + x^2 < \frac{3\pi}{2}.$$

This is solved by

$$\sqrt{\frac{\pi}{2} - 1} < x < \sqrt{\frac{3\pi}{2} - 1}.$$

But no matter, there is an interval where the function y is well-defined as a function of x, and here the Leibniz Rule gives us

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}\left[\int_{1+x^2}^{2} \tan t \; dt\right] = 0 - \tan\left(1 + x^2\right) \cdot 2x = -2x\tan\left(1 + x^2\right),$$

for x on the above domain.

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