**Name:** ______________________________ **Section Number:** ____________

# 110.106 CALCULUS I (Biology & Social Sciences)
# FALL 2009
# MIDTERM EXAMINATION SOLUTIONS
# December 4, 2009

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**Instructions:** The exam is **8** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

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## PLEASE DO NOT WRITE ON THIS TABLE !!

| Problem | Score | Points for the Problem |
|---|---|---|
| 1 | | 20 |
| 2 | | 15 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 30 |
| 6 | | 15 |
| TOTAL | | 100 |

### Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

**Signature:** ______________________________ **Date:** ____________

**Question 1.** [20 points] Do the following:

**(a)** Determine if the function $h(x) = \dfrac{(\ln x)^2}{x^2}$ has a horizontal asymptote.

**Solution:** $h(x)$ has a horizontal asymptote if either $\lim_{x\to\infty} h(x)$ or $\lim_{x\to-\infty} h(x)$ exist. We check the first: Note that $h(x)$ is differentiable on the interval $(0,\infty)$, and both $(\ln x)^2$ and $x^2$ go to infinity as $x$ goes to infinity. Hence this limit $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x^2}$ is in the indeterminate form $\dfrac{\infty}{\infty}$. Hence by L'Hospital's Rule

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{(\ln x)^2}{x^2} = \lim_{x\to\infty} \frac{2(\ln x)\left(\frac{1}{x}\right)}{2x} = \lim_{x\to\infty} \frac{\ln x}{x^2}.$$

This last limit is again in the indeterminate form $\dfrac{\infty}{\infty}$, so we use L'Hospital's Rule again, to get

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{2x} = \lim_{x\to\infty} \frac{1}{2x^2}.$$

From Chapter 1, we know that $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$. Hence $\lim_{x\to\infty} \dfrac{1}{2x^2} = 0$ and $\lim_{x\to\infty} h(x) = 0$ so that $y = 0$ is a horizontal asymptote of $h(x)$.

**(b)** Find $f'(x)$, for $f(x) = x^{\sin x}$.

**Solution:** Rewrite $f(x)$ as

$$x^{\sin x} = e^{\ln x^{\sin x}} = e^{(sinx)(\ln x)}.$$

Recall that for any differentiable function $g(x)$, $\dfrac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)$. So in this case, $g(x) = (\sin x)(\ln x)$, so

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[e^{(\sin x)(\ln x)}\right] \\ &= e^{(\sin x)(\ln x)} \cdot \frac{d}{dx}\left[(\sin x)(\ln x)\right] \\ &= e^{(\sin x)(\ln x)} \left((\cos x)(\ln x) + (\sin x)\frac{1}{x}\right) \\ &= x^{\sin x}\left((\cos x)(\ln x) + \frac{sinx}{x}\right). \end{aligned}$$

**Question 2.** [15 points] For the function $g(x) = 4x^5 - 5x^4$ defined over all $\mathbb{R}$, do the following:

**(a)** Find all critical points and determine whether they are local minima, local maxima, or neither.

**Solution:** The critical points are, by definition, the points in the domain where either $g'(x) = 0$ or $g'(x)$ is not defined. Since $g(x)$ is a polynomial defined on $\mathbb{R}$, the only critical points are the solutions to $g'(x) = 0$. Here

$$g'(x) = 20x^4 - 20x^3 = 20x^3(x-1)$$

which is solved by $x = 0$ and $x = 1$. Using the Second Derivative Test, $g'' = 80x^3 - 60x^2 = 20x^2(4x-3)$, and $g''(0) = 0$ and $g''(1) > 0$. Hence, $x = 1$ is a local minimum, but the Second Derivative Test is inconclusive for $x = 0$. Fortunately, we know that the first derivative $g'(x) = 20x^3(x-1)$ is positive for $x < 0$ and negative for small $x > 0$. Hence it must be the case that $x = 0$ is a local max.

**(b)** Find all inflection points and determine the concavity of $g(x)$.

**Solution:** The inflection points are, by definition, the points in the domain where $g''(x) = 0$ or $g''(x)$ is not defined, and where the concavity of $g(x)$ changes. The concavity of $g(x)$ is determined solely by the sign of $g''(x)$: $g(x)$ is concave up where $g''(x) > 0$, and down where $g''(x) < 0$. Again, $g(x)$ is a polynomial defined on $\mathbb{R}$, so $g''(x)$ is defined everywhere. In part **(a)** above, $g'' = 80x^3 - 60x^2 = 20x^2(4x-3)$, and $g''(0) = 0$ is solved by $x = 0$ and $x = \frac{3}{4}$. On the interval $(-\infty, 0)$, $g''(x) < 0$, so $g(x)$ is concave down. On the interval $\left(0, \frac{3}{4}\right)$, $g''(x) < 0$, and $g(x)$ continues to be concave down. And on the interval $\left(\frac{3}{4}, \infty\right)$, $g''(x) > 0$. Here $g(x)$ is concave up. Hence, $x = \frac{3}{4}$ is the only inflection point.

**(c)** Find all vertical and horizontal asymptotes, if any.

**Solution:** The shortest answer is: There aren't any. The briefest justification is that polynomials of any positive degree do not have asymptotes of either type.

**Question 3.** [10 points] Let $f(x)$ be any differentiable function on $\mathbb{R}$. If $f(1) = 2$ and $f(3) = 7$, show that it must be the case that somewhere in the interval $(1, 3)$, the derivative of $f$ is greater than 2.

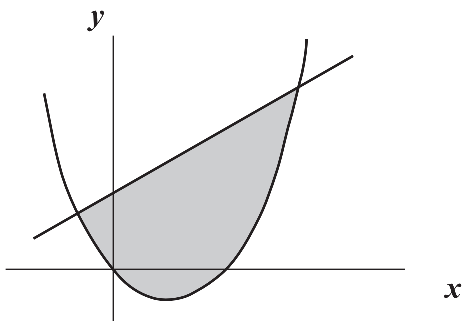
**Solution:** Since $f(x)$ is differentiable everywhere, it is continuous everywhere. Hence, $f(x)$ is differentiable on $(1, 3)$ and continuous on $[1, 3]$. By the Mean Value Theorem, there must exist a point $c \in (1, 3)$ where

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{7 - 2}{3 - 1} = \frac{5}{2}.$$

But then since $\frac{5}{2} > 2$, there must exist a point $c$ where its derivative is greater than 2.

**Question 4.** [10 points] Find the area of the region enclosed by the functions $y = x^2 - 2x$ and $y = x + 4$.

**Solution:** Using the graphs at left, the shape "begins" and "ends" at the corners defined by the intersections




of the line $y = x+4$ and the parabola $y = x^2 - 2x$. These intersection points are the solutions to the equation $x + 4 = x^2 - 2x$. Here

$$\begin{aligned} x + 4 &= x^2 - 2x \\ 0 &= x^2 - 3x - 4 = (x-1)(x+1) \end{aligned}$$

which is solved by $x = -1$ and $x = 4$. Let $f(x) = x + 4$ and $g(x) = x^2 - 2x$, and recall that the area between the graphs of $f(x)$ and $g(x0)$, when $f(x) \geq g(x)$ between $x = a$ and $x = b$ is $\int_a^b (f(x) - g(x))\, dx$. Here, then

$$\begin{aligned} \int_{-1}^{4} \left[(x+4) - (x^2 - 2x)\right]\, dx &= \int_{-1} 4(-x^2 + 3x + 4)\, dx \\ &= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x\right)\Big|_{-1}^{4} = -\frac{64}{3} + \frac{48}{2} + 16 - \frac{1}{3} - \frac{3}{2} + 4 = 20\frac{5}{6}. \end{aligned}$$

**Question 5.** [30 points] Evaluate the following:

**(a)** $\int \left( \frac{1}{2+x} + e^{3x} - \sin(x+1) \right) dx.$

**Solution:** All three of these can be integrated without directly doing the substitution. Directly

$$\int \frac{1}{x+2} + e^{3x} - sin(x+1)\, dx = \ln|x+2| + \frac{1}{3}e^{3x} + \cos(x+1) + C.$$

**(b)** $\frac{d}{dx}\left[\int_{x^2+x}^{0} e^x \sin x\, dx\right].$

**Solution:** By the Leibniz Rule,

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(x)\, dx)\right] = f\left[h(x)\right] h'(x) - f\left[g(x)\right] g'(x).$$

In this case, $h(x) = 0$. Hence the first term on the right-hand-side is 0. Also, $g(x) = x^2 + x$, $f(x) = e^x \sin x$, and $g'(x) = 2x + 1$. Putting these together, we get immediately

$$\frac{d}{dx}\left[\int_{x^2+x}^{0} e^x \sin x\, dx\right] = -e^{(x^2+x)} \sin(x^x + x) \cdot (2x+1).$$

**(c)** $\int_2^5 \frac{3x}{\sqrt{x-1}}\, dx$ (Hint: Use the substitution $u = x - 1$).

**Solution:** Here the trick, using the substitution, is to make sure that ALL of the $x$'s are changed into $u$'s. With the substitution $u = x - 1$, we get $du = dx$, and $u + 1 = x$. Also, for the limits, when $x = 2$, $u = 1$, and when $x = 5$, $u = 4$. Thus,

$$\int_2^5 \frac{3x}{\sqrt{x-1}}\, dx = 3\int_2^5 \frac{x}{\sqrt{x-1}}\, dx = 3\int_1^4 \frac{u+1}{\sqrt{u}}\, du = 3\int_1^4 \left(\frac{u}{\sqrt{u}} + \frac{1}{sqrtu}\right) du = 3\int_2^5 \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du.$$

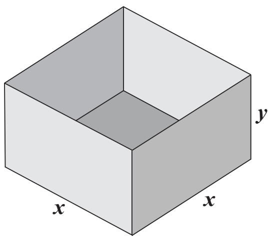
The integrand now is a sum of power functions, and hence only the anti-power rule is needed:

$$3\int_2^5 \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du = 3\left(\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right)\Bigg|_1^4 = 3\left(\frac{2}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}}\right) - 3\left(\frac{2}{3}(1)^{\frac{3}{2}} + 2(1)^{\frac{1}{2}}\right) = 16 + 12 - 2 - 6 = 20.$$

**Question 6.** [15 points] Do EXACTLY ONE of the following:

**(a)** A small box with a square base and an open top must enclose a volume of 32 cm$^3$. Find the dimensions of the box that minimizes the amount of material needed to make it. (Hint: Minimize the surface area.)

**Solution:** The value of a box with sides as in the figure is $V = x^2y$, and its surface area is $S = 4xy+x^2$




(there are four sides and a bottom, no top). Here we want to minimize the surface area, but $S$ is a function of two variables. Fortunately, we have the volume constraint $32 = x^2y$. Solving for $y$, we get $y = \frac{32}{x^2}$. Subbing this into the surface area expression, we get

$$S = 4x\left(\frac{32}{x^2}\right) + x^2 = \frac{128}{x} + x^2.$$

Here, $S$ is differentiable on its domain $(0,\infty)$, so the only critical points are where $S'(x) = 0$:

$$S'(x) = -\frac{128}{x^2} + 2x.$$

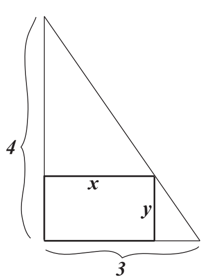
This is solved only by $x = 4$ (work this out). To see what kind of critical point $x = 4$ is, use the Second Derivative Test:

$$S''(x) = \frac{256}{x^3} + 2 > 0 \text{ on all of } (0,\infty).$$

Hence $x = 4$ is a local minimum (a global minimum, actually) (Why?). When $x = 4$, we have $y = \frac{32}{4^2} = 2$, and the dimensions of the box which minimizes surface area is $x = 4$ and $y = 2$.

**(b)** Find the dimensions of the largest area rectangle that can be placed inside the right triangle below where the rectangle shares the right angle with the triangle. (Hint: The rectangle will touch the hypotenuse with its opposite corner, and you will need to use similar triangles in part of this problem.)

**Solution:** Here, we seek to maximize the area $A$ of the rectangle, where $A = xy$. To rewrite this as a




function of only one variable, we note that both the triangle sitting on top of the rectangle, as well as the one to the right, are both similar to the whole triangle. Similar triangles have proportional sides. If we choose the top one, we note that it has a base of 4 and a height of $4-y$ (can you see this?) Hence

$$\frac{\text{base}}{\text{height}} = \frac{3}{4} = \frac{x}{4-y}$$

which simplifies to $x = \frac{3}{4}(4-y)$. Sub this into the area equation and

$$A = xy = \frac{3}{4}(4-y)y = 3y - \frac{3}{4}y^2.$$

Here, area is non-negative on $[0,4]$ (at these two extremes, the rectangle will have 0 area, since one of its sides has 0 length, Can you see this?). It's only critical point is at $A'(y) = 3 = \frac{3}{2}y$ which is solved by $y = 2$. We also know that this corresponds to a maximum area, since $A''(2) = \frac{3}{2} < 0$. And when $y = 2$, $x = \frac{3}{4}(4-2) = \frac{3}{2}$. These are the dimensions of the rectangle in the drawing that maximizes area.