Name: ____

110.106 CALCULUS I (Biology & Social Sciences)
FALL 2009
MIDTERM EXAMINATION SOLUTIONS
October 12, 2009

Instructions: The exam is **8** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		20
2		20
3		8
4		8
5		16
6		18
7		10
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:**______________

Question 1. [20 points] Evaluate the following:

(a) $\displaystyle\lim_{x\to-\infty}\frac{3x-4x^3}{2x^3+6x^2}$.

Solution: Really, one can appeal directly to the degree theorem for limits at infinity of rational functions that we did in class. Indeed, noting that $f(x)=\dfrac{3x-4x^3}{2x^3+6x^2}$ is a rational function, where the degree of the numerator and the denominator are both 3. Thus, the limit at either infinity or negative infinity is simply the ratio of the leading coefficients. This would be enough to establish that

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{3x-4x^3}{2x^3+6x^2}=\frac{-4}{2}=-2.$$

You could actually calculate the limit, via multiplying by the "clever form of 1", so that

$$\lim_{x\to-\infty}\frac{3x-4x^3}{2x^3+6x^2}=\lim_{x\to-\infty}\frac{3x-4x^3}{2x^3+6x^2}\cdot\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)=\lim_{x\to-\infty}\frac{\frac{3}{x^2}-4}{2+\frac{6}{x}}$$

and appealing to the fact that

$$\lim_{x\to-\infty}\frac{\frac{3}{x^2}-4}{2+\frac{6}{x}}=\frac{\displaystyle\lim_{x\to-\infty}\frac{3}{x^2}-4}{\displaystyle\lim_{x\to-\infty}2+\frac{6}{x}}=\frac{\left(\displaystyle\lim_{x\to-\infty}\frac{3}{x^2}\right)-4}{2+\left(\displaystyle\lim_{x\to-\infty}\frac{6}{x}\right)}=\frac{0-4}{2+0}=-2.$$

But simply mentioning the theorem and the result will work, in this case, since we did the theorem in class.

(b) $\displaystyle\lim_{x\to0}\frac{\sin(4x)}{-x}$.

Solution: Note that this problem looks a lot like the limit at 0 of the function $\dfrac{\sin x}{x}$, except for the coefficient of x inside the sine function. To account for this distinction, consider the following change of variable, $z=4x$. This is like problem 6, Section 3.4, and like one that we did in lecture. Knowing that as $x\to0$, also $z\to0$, we get

$$\lim_{x\to0}\frac{\sin(4x)}{-x}=\lim_{z\to0}\frac{\sin z}{-\left(\frac{z}{4}\right)}=\frac{1}{-\left(\frac{1}{4}\right)}\cdot\left(\lim_{z\to0}\frac{\sin z}{z}\right)=-4\cdot1=-4.$$

Question 2. [20 points] Evaluate the following:

(a) $\lim_{n\to\infty} a_n$, For $a_n = \dfrac{1+\sin n}{n^2}$.

Solution: There are actually two interesting ways to do this problem. One can appeal to the Sandwich Theorem, or pick apart the function into pieces. Here are both:

Sandwich Theorem: Like the limit in the previous problem, this problem looks a lot like the limit at ∞ of the function $\dfrac{\sin x}{x}$. If you recall, we used the Sandwich Theorem to calculate that limit. We can use the example from the lecture as a guide here, or we can use Problem 2, Section 3.4 as a guide also. Note that $-1 \leq \sin n \leq 1$ for any natural number n. But then by adding 1 to each of the parts of this inequality,

$$1 - 1 \leq 1 + \sin n \leq 1 + 1,$$

so that $0 \leq 1 + \sin n \leq 2$ for all natural numbers n. Dividing by n^2 for any positive choice of n, we get

$$0 = \frac{0}{n^2} \leq \frac{1+\sin n}{n^2} \leq \frac{2}{n^2}.$$

This inequality allows us to use the Sandwich Theorem effectively. Since for all positive natural numbers n, the function $\dfrac{1+\sin n}{n^2}$ is bounded above by $\dfrac{2}{n^2}$ and below by 0, we know by the Sandwich Theorem that

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{1+\sin n}{n^2} \leq \lim_{n\to\infty} \frac{2}{n^2}.$$

Since both the left hand and the right hand limits are 0, it follows that

$$\lim_{n\to\infty} \frac{1+\sin n}{n^2} = 0.$$

Other way: Notice that

$$\frac{1+\sin n}{n^2} = \frac{1}{n^2} + \frac{\sin n}{n^2} = \frac{1}{n^2} + \frac{1}{n}\cdot\frac{\sin n}{n}.$$

Then

$$\lim_{n\to\infty} \frac{1+\sin n}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{1}{n}\cdot\frac{\sin n}{n}\right) = \left(\lim_{n\to\infty}\frac{1}{n^2}\right) + \left(\lim_{n\to\infty}\frac{1}{n}\right)\cdot\left(\lim_{n\to\infty}\frac{\sin n}{n}\right),$$

as long as each of the individual limits on the right-hand-side of the last equality all exist. But they all do. In fact, each one of them is 0. The first is 0 since

$$\lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{n} = \left(\lim_{n\to\infty}\frac{1}{n}\right)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right) = 0\cdot 0 = 0.$$

And the last limit we calculated in class. Hence all limits exist and again

$$\lim_{n\to\infty} \frac{1+\sin n}{n^2} = 0.$$

(b) Assuming $\lim_{n\to\infty} b_n$ exists for the recursively defined sequence

$$b_{n+1} = \frac{5}{2}b_n(1-b_n),$$

What are the only possible values for the limit (that is, what are the fixed points of $\{b_n\}$)?

Solution: To find the fixed points of this sequence, simply look for values where the output at any stage n, equals the input. Thus look for the values of b that satisfy

$$b = \frac{5}{2}b(1-b).$$

Not this means that we are looking for those sequences $\{b_n\}$ where $b_{n+1} = b_n$ for all choices of n. Here

$$\begin{aligned} b &= \frac{5}{2}b - \frac{5}{2}b^2, \\ 0 &= \frac{3}{2}b - \frac{5}{2}b^2, \\ 0 &= \frac{1}{2}b\left(3 - 5b\right), \end{aligned}$$

which is solved by $b = 0$ and $b = \dfrac{3}{5}$. These are the fixed points, and the only possible values can can serve as a limit for a sequence $\{b_n\}$ that satisfies this recursion. The model for this problem is the HW Problems 107 and 108 in Section 2.2.

Question 3. [8 points] Find a value of the constant c so that the function $f(x)$ is continuous at $x = 2$, where

$$f(x) = \begin{cases} x^2 + cx + 1 & x < 2 \\ \frac{8}{x^2} & x \geq 2 \end{cases} .$$

Solution: Note that the two functions defining $f(x)$ here are continuous on their respective domains. One piece is a rational function, continuous on the closed interval $[2, \infty)$, and the other is a polynomial for any choice of c on the interval $(-\infty, -2)$. In fact, we can make the polynomial continuous on $(-\infty, 2]$ by assigning it the value of the lower-side limit at 2. In this case, this is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 + cx + 1 = 2^2 + 2c + 1 = 5 + 2c$$

for a choice of c. Hence it remains only to find a value of c so that the above limit equals the actual current function value at 2: $f(2) = \dfrac{8}{2^2} = 2$. To this end, we solve $5 + 2c = 2$, so that $c = -\dfrac{3}{2}$. This problem is a lot like the HW Problem 28 in Section 3.2.

Question 4. [8 points] Show $h(x) = x^7 + 5x^3 - 1$ has a root in the interval $[0, 1]$.

Solution: This problem is an immediate and quick use of the Intermediate Value Theorem (IVT). The function $h(x)$ is a polynomial so is continuous on the domain $[0, 1]$. At the end points of the interval, we have

$$\begin{aligned} h(0) &= 0^7 + 5 \cdot 0^3 - 1 = -1 < 0, \\ h(1) &= 1^7 + 5 \cdot 1^3 - 1 = 5 > 0. \end{aligned}$$

Since h is continuous, and $h(0) < 0 < h(1)$, it follows by the IVT that there exists a value $c \in (0, 1)$, where $h(c) = 0$. This value c is a root of the polynomial $h(x)$. See Problem 2 of the HW in Section 3.5.

Question 5. [16 points] Let $f(x) = \dfrac{\sqrt{x^2+6x}}{3x}$. Do the following:

(a) Find the domain of f.

Solution: The function $f(x)$ will be continuous as long as the denominator is not 0 and the expression under the radical is non-negative. To this end, you can solve for these points separately. First, $x = 0$ cannot be in the domain of this function, since the denominator is 0 here. Second, only valid domain points will solve the inequality

$$x^2 + 6x \geq 0.$$

But this factors quickly in to the pieces

$$x(x+6) \geq 0.$$

This last inequality is solved by all x, where $x \geq 0$ and $x \leq -6$. Putting these two calculations together, we get

$$\text{the domain of } f(x) = \left\{ x \in \mathbb{R} \mid x \leq -6 \text{ or } x > 0 \right\}.$$

(b) Calculate $f'(x)$.

Solution: A good way to start is to recognize that this is a quotient of two functions. Hence, the quotient rule will work here, and

$$f'(x) = \frac{d}{dx}\left(\frac{\sqrt{x^2+6x}}{3x}\right) = \frac{\left(\frac{d}{dx}\sqrt{x^2+6x}\right)\cdot 3x - \sqrt{x^2+6x}\cdot\left(\frac{d}{dx}3x\right)}{(3x)^2}.$$

This works fine, except that the first derivative you will have to take is that of a composite function, and hence the chain rule is needed. Let $g(x) = \sqrt{x}$ and $h(x) = x^2 + 6x$. Then

$$(g \circ h)(x) = g(h(x)) = g\left(x^2 + 6x\right) = \sqrt{x^2+6x}.$$

And

$$\frac{d}{dx}\sqrt{x^2+6x} = \frac{d}{dx}\left[g(h(x))\right] = g'(h(x)) \cdot h'(x) = \frac{1}{2\sqrt{x^2+6x}} \cdot (2x+6) = \frac{2x+6}{2\sqrt{x^2+6x}}.$$

Now throw this into the above calculation (and calculate the derivative of $3x$ also) to get

$$f'(x) = \frac{\left(\frac{2x+6}{2\sqrt{x^2+6x}}\right)\cdot 3x - \sqrt{x^2+6x}\cdot(3)}{(3x)^2}.$$

Best to stop at this point and not simplify.

Question 6. [18 points] Let $g(x) = 2x^2 - 3$. Do the following:

(a) Use the definition of the derivative to show that $g'(2) = 8$.

Solution: Let's just follow the two main definitions: Either

$$\begin{aligned} g'(2) = \lim_{x\to 2} \frac{g(x)-g(2)}{x-2} &= \lim_{x\to 2} \frac{2x^2-3-5}{x-2} = \lim_{x\to 2} \frac{2x^2-8}{x-2} = \lim_{x\to 2} \frac{2(x^2-4)}{x-2} \\ &= \lim_{x\to 2} \frac{2(x+2)(x-2)}{x-2} = \lim_{x\to 2} 2(x+2), \end{aligned}$$

where we can cancel out the common factor $(x-2)$ in the fraction. What is left is continuous at $x = 2$, hence $g'(2) = 2 \cdot 4 = 8$. Or

$$\begin{aligned} g'(2) = \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} &= \lim_{h\to 0} \frac{2(2+h)^2-3-5}{h} = \lim_{h\to 0} \frac{2(2^2+4h+h^2)-8}{h} \\ &= \lim_{h\to 0} \frac{8h-2h^2}{h} = \lim_{h\to 0} \frac{h(8-2h)}{h} = \lim_{h\to 0} 8-2h = 8. \end{aligned}$$

Or

$$\begin{aligned} g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} &= \lim_{h\to 0} \frac{2(x+h)^2-3-(2x^2-3)}{h} = \lim_{h\to 0} \frac{2(x^2+2xh+h^2)-2x^2+3}{h} \\ &= \lim_{h\to 0} \frac{4xh-2h^2}{h} = \lim_{h\to 0} \frac{h(4x-2h)}{h} = \lim_{h\to 0} 4x-2h = 4x, \end{aligned}$$

and since we seek the derivative at $x = 2$, we get $g'(2) = 4 \cdot 2 = 8$. Any of these will work. We did a problem much like this in class.

(b) Find the equation of the line tangent to $g(x)$ at $x = 2$.

Solution: The formula for the tangent line to $g(x)$ at $x = 2$ is

$$y - g(2) = g'(2)(x-2).$$

Here, $g(2) = 5$ and $g'(2) = 8$. So

$$y - 5 = 8(x-2).$$

(c) For $f(x) = \dfrac{1}{\sqrt{x}}$, find $(g \circ f)(x)$ and specify its domain.

Solution: Here

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{\sqrt{x}}\right)$$

where $g(x) = 2x^2 - 3$. So

$$(g \circ f)(x) = 2\left(\frac{1}{\sqrt{x}}\right)^2 - 3 = \frac{2}{x} - 3,$$

but there is really no need to simplify the function to the last step. The domain is most easily seen without the simplification. The domain for the inside function f is $x > 0$, and since the outside function $g(x)$ is defined for all input values, it follows that the domain for $(g \circ f)(x)$ is also all $x > 0$. We have done a number of these in class, and HW Problem 16, Section 1.2 is very close to this one.

Question 7. [10 points] Do exactly ONE of the following:

(a) Find the slope of the line tangent to the curve given by the equation $y^2 = x^2 + xy$ at the point $(-1, 0)$.

Solution: Since the equation ties together the variables x and y, we can view y as an implicit function of x. Then both sides of the equation may be thought of as functions of x. Since they are equal, their derivatives will be equal also, and

$$\begin{aligned} \frac{d}{dx}\left[y^2\right] &= \frac{d}{dx}\left[x^2 + xy\right] \\ 2y\frac{dy}{dx} &= 2x + 1 \cdot y + x \cdot \frac{dy}{dx} = 2x + y + x\frac{dy}{dx}. \end{aligned}$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{2x+y}{2y-x}.$$

Evaluated at the point $(-1, 0)$, we get

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1 \\ y=0}} = \left.\frac{2x+y}{2y-x}\right|_{\substack{x=-1 \\ y=0}} = \frac{2(-1)+0}{2(0)-(-1)} = -2.$$

Number 58, Section 4.4 in your HW is a good example of this type of problem. Special note: There is a problem with this problem as stated. can you see it?

(b) The volume of a spherical balloon is $V = \frac{4}{3}\pi r^3$ where the radius r is measured in centimeters. If the volume is expanding at a constant rate of 100 $\frac{\text{cm}^3}{\text{sec}}$, how fast is the radius expanding when the radius is exactly 10 cm?

Solution: Due to the equation relating volume and radius, if one variable is changing in time, so is the other. Since the volume is changing at a constant rate, the function $V(t)$ is differentiable, and hence so it $r(t)$. And since the two sides of the volume equation are equal, so are their derivatives, and

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt}.$$

We know that at the point where $r(t) = 10$ cm, $\frac{dV}{dt} = 100\ \frac{\text{cm}^3}{\text{sec}}$, so

$$100 = \frac{4}{3}\pi\left(3(10)^2\right) \cdot \frac{dr}{dt},$$

and hence $\frac{dr}{dt} = \frac{1}{4\pi}\ \frac{\text{cm}}{\text{sec}}$. HW Problem 70 of Section 4.4 is a more involved version of this type of problem.