110.108 CALCULUS I

Week 8 Lecture Notes: October 17 - October 21

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1. Lecture 1: The Extreme Value Theorem

A while back, we talked about the Intermediate Value Theorem.

(Extreme Value Theorem (EVT)). *For a function f continuous on an interval $[a, b]$, where $f(a) \neq f(b)$, for every $f(a) \leq y \leq f(b)$, or $f(b) \leq y \leq f(a)$ (whichever the case may be), there is an $c \in (a, b)$ where $f(c) = y$.*

The basic idea is that if a function is continuous, and achieves certain values at two points $x = a$ and $x = b$, then $f(x)$ must achieve every value in between. This is a nice general principle useful in the general sense of how continuous functions work. The idea that one can always draw the graph of a continuous function on an interval without "lifting the pencil from the paper" is encoded in this theorem. The idea that every horizontal line between the values of $f(a)$ and $f(b)$ must intersect the graph at least once is a restatement of this tenet. If there is a shortcoming to this theorem, it is the fact that it is an example of an "existence theorem": A theorem that establishes that something exists without ANY idea as to how to locate it. Of course it is always good to know something exists before setting out to find it (think of the Holy Grail, the Loch Ness monster, or the Higgs Boson as examples of looking for things that may in fact not exist at all!). Hence the importance of this theorem is still high, even if it lacks direct utility in calculations.

Today, we introduce another existence theorem, and then explore what will become a technique for actually finding the object we are seeking. This one is called the *Extreme Value Theorem*:

[Fermat's Theorem] Let $f(x)$ be continuous on a closed interval $[a, b]$. Then f achieves its absolute maximum and its absolute minimum on $[a, b]$.

Notes:

- This theorem is too hard to prove at this level, and requires a much deeper understanding of just how subsets of the real line and functions work. Unfortunately, that will come in a course like 110.405 Analysis or above. We can understand how and why it works, however, at this level.
- It should seem intuitively obvious that the theorem is true, but the subtleties of just why each of the suppositions are necessary is always a good way to better understand the principle.
- First, notice that I didn't simply stop and say the the interval is closed. I actually specified that the interval is closed of the form $[a, b]$. That means that I cannot use an interval like $(-\infty, 1]$ as my closed interval. The function $f(x) = x^2$ is continuous on $(-\infty, 1]$, and yet does not have a maximum there (why not?). When the theorem says closed and specifies that the interval is of the form $[a, b]$, it means that a and b MUST be real numbers. Thus the interval is bounded also.
- That the interval must be closed also is necessary. If the theorem allowed open intervals like $(0, 2)$, then a function like $g(x) = \ln x$ is continuous on the interval, but does NOT have a minimum there (remember the graph?). It is the asymptote at one of the endpoints that makes the function fail to have one of the extrema.

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- If the function is not continuous on the interval, then the theorem is also not true. As an example, try to fix the function $g(x) = \ln x$ on $[0, 2]$ by defining the value at $x = 0$. Define
$$g(x) = \begin{cases} 2 & x = 0 \\ \ln x & 0 < x \leq 2. \end{cases}$$
Actually, we can define $g(0)$ as ANY real number. The new function will not be continuous at $x = 0$ and the function $g(x)$ will still not have a minimum on $[0, 2]$.
- With the closed, bounded interval and a continuous function, we are assured that the function has a maximum and minimum value on the interval. Existence is a certainty. Now how to find such values and the points where that evaluate to.

We turn to a nice start, courtesy of Pierre Fermat:

Theorem 1 (Fermat's Theorem)**.** *If $f(x)$ has a local extremum at a point $x = c$, and $f'(c)$ exists, then it must be the case that $f'(c) = 0$.*

This follows the discussion in class that, for "nice" functions (defined here as those functions that have derivatives), it should be possible to locate some possibilities for extrema by finding places where the tangent line is horizontal (where the derivative is 0, or as we say, where the derivative *vanishes*). It turns out that by this theorem, this is indeed the case.

Proof. Suppose for now that f has a local maximum at $x = c$. By definition, this means that for all points x near c (read: in a small open interval containing c), we know that $f(x) \leq f(c)$. To make this a bit more precise, choose a small positive $h > 0$, small enough that for all $x \in [c, c + h)$, $f(c + h) \leq f(c)$. We are only working to the right of c at the moment, and since $f(c)$ is a local max, this is possible. Now since $f(c + h) \leq f(c)$, we know also that
$$f(c+h) - f(c) \leq 0, \quad \text{and} \quad \frac{f(c+h) - f(c)}{h} \leq \frac{0}{h} = 0,$$
since h is positive (the sense does not change in the equality by division with a positive number). For all different values of $h > 0$ this remains true, and hence by Theorem 2.3.2 in Chapter 2 (thinking of each side as a function of h), we also know that
$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} = 0.$$
But the left hand side of the inequality is precisely the right-side limit for the definition of $f'(c)$, which we know exists by supposition (and hence must equal its side limits!). Hence this last statement says that the derivative of f at $x = c$ cannot be positive, and
$$f'(c) \leq 0.$$

Now let $h < 0$ be a small negative number (and hence we will work on the left side of $x = c$. Again, since $f(c)$ is a local maximum, if we choose h small enough, we know that $f(c + h) \leq f(c)$. Note that this looks exactly the same, yet it is not since we have $h < 0$ here. Why this is different is that here, we see that
$$f(c+h) - f(c) \leq 0, \quad \text{while} \quad \frac{f(c+h) - f(c)}{h} \geq \frac{0}{h} = 0.$$

Why this is true should be elementary: When we divide by a small negative number h, we change the sense of the inequality, right? Following above (and reusing Theorem 2.3.2), we then see that now

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq \lim_{h\to 0^-} = 0.$$

The left hand side of the inequality is now the left-side limit of $f'(c)$ (which exists and must equal both its side limits). But this last statement says that the derivative of f at $x = c$ cannot be negative, and

$$f'(c) \geq 0.$$

For $f'(c)$ to satisfy both of these inequalities, it can only be the case that $f'(c) = 0$. Note that the case for the local extremum being a minimum is entirely similar. □

Some Final Notes:

- If you think of what the graph of a local maximum looks like, the whole notion of the derivative (as the slope of the tangent line) as a limit of the slopes of secant line along the curve should make sense here. On the right, the function should dip down (or stay the same height. The the slopes of ALL of the secant lines should dip down (or stay the same height). Thus the limit should either dip down or stay the same height (slope 0). On the left, the slopes of ALL of the secant lines should either be positive or 0, hence this should also be the case in the limit. Think about this and draw pictures. You will see it soon enough (if you do not already).
- Notice that we did not specify whether the interval is closed or not in this theorem. One can conclude that it is not necessary to include this. Why not? Suppose the interval we define $f(x)$ on is closed (includes the endpoint). Then it is certainly possible that the local maximum can be AT the endpoint. But remember that we do not define derivatives at edges of intervals. When we speak of a differentiable function on a closed interval $[a, b]$, we mean a function that is (1) continuous on $[a, b]$, and differentiable on (a, b). Thus the possible extrema at eh endpoints do not satisfy the supposition that f' exists there. Thus the theorem is safe even for this situation. We just cannot apply the theorem to end point extrema.
- The converse of Fermat's Theorem is certainly NOT true! IN other words, there exist points c where $f'(c) = 0$, but f does have a local extrema at $x = c$. As an example, think $g(x) = x^3$. Here $g'(0) = 0$, but everything to the right of $g(0)$ is higher than 0, while everything to the left is lower then 0.
- $f(x)$ can have a local extremum at $x = c$ and $f'(c)$ need not exist. AS an example, try $h(x) = -|x| + 2$. Here, $x = 0$ is the point where $h(x)$ achieves its global maximum of 2. But $h'(0)$ does not exist (why not?)

So now we have some idea that extrema may exist at places where the derivative vanishes (equals 0). Or they may exist at places where the derivative does not exist. How about other places? First, let's try to legitimize both of these possibilities into one type of set of special points:

Definition 2. A *critical point* of $f(x)$ is a point $x = c$ in the domain where either $f'(c) = 0$, or $f'(c)$ does not exist.

As a matter of making sure we are speaking the same language, keep in mind the following: Local and absolute extrema (maxima and minima) are *values* of the function, and are part of the range (output) of a function like $f(x)$. On the other hand, points (like the critical points above) are input values, x-values, or domain values. When we talk of a function's absolute maximum, it is the value of $f(c)$ at the point $x = c$. For example, the absolute maximum of $f(x) = x^2 + 2$ on the interval $[-1, 3]$ is 11. It is located at the right endpoint $x = 3$. The other extrema of $f(x)$? $f(x)$ also has an absolute minimum of 2, located at $x = 0$ and a local maximum of 3, located at $x = -1$.

Notes:

- Points are called critical because they are special; something about the function has the potential to change here. It may stop rising and start falling. It may change its direction suddenly (a corner of the graph). The function may jump discontinuously at a point, or stop there. All of these changes in behavior would be of interest to someone using a function to model behavior or study how one entity's values are changing with respect to a known entity.
- Keep in mind that a critical point MUST be in the domain of the function. The definition only makes sense for points where the function is defined.
- How can a function be continuous and yet at a point $x = c$, $f'(c)$ not be defined?
  - at the corner of a graph,
  - at a break in the graph (if the function is not continuous, of course), and
  - at a place where the tangent line is vertical. Here, the tangent line is defined, and yet the derivative (its slope) is not). As an example, think $f(x) = \sqrt[3]{x}$ and look at $x = 0$.
- Here is a good fact: If $f(x)$ has a local extremum at $x = c$, then c must be a critical point of f. Given this, while not every critical point is extreme, it is at the critical points where the local extrema will always be found. This is crucial!

Thus we can now produce an easy (to state) procedure for find the absolute extrema of any continuous function defined on a closed, bounded interval: This is the *Closed Interval Method*on page 278. For $f(x)$ continuous on $[a, b]$, the absolute maximum and minimum values of f, assured by the EVT can be found by doing the following:

(1) Locate all critical points in the open interval (a, b). (a) Set $f'(x) = 0$ and solve for all points x, and (b) locate all points where $f'(x)$ is not defined.
(2) Evaluate f on all critical points and the two endpoints, and
(3) Choose the maximum and minimum values from this list.

Remember that the highest and lowest *values* of the function are the absolute extrema and they are definitely unique! It may certainly be that there are many points that evaluate to these extreme values (think of every point on a horizontal line, or all of the peaks of the sine function on a large domain), but the actual function value which is the max will be the only one.

Example 3. Find the absolute maximum and minimum of the function $f(x) = \ln x^2 + x + 1$ on the interval $[-1, 1]$.

Strategy: Since we do not have a good intuition of what this function looks like (well, if you look below, we actually do, but let's fly blind for the moment), we must rely on the analysis given by the above discussion to locate the highest and lowest values of this function. We will use the above procedure exactly and solve this by the numbers:

Solution:

First, let's find all critical points in $(-1,1)$: Note that $f(x)$ is a composite function, and the inside function x^2+x+1 is a polynomial which is continuous everywhere (polynomials always are!). Also, x^2+x+1 is never 0, and since it takes the value 1 when we input $x=0$, it is never negative either (using the quadratic formula to try to solve $x^2+x+1=0$, we find that it has no real solutions). Hence $f(x)$ will be continuous on all of $[-1,1]$. Its derivative is

$$f'(x) = \frac{d}{dx}\left[\ln x^2+x+1\right] = \frac{2x+1}{x^2+x+1},$$

which is a rational function whose denominator is never 0. Hence the derivative is defined on all of $(-1,1)$. Hence the ONLY critical points are the places where $f'(x)=0$. This is only where $2x+1=0$, or when $x=-\frac{1}{2}$. Thus the ONLY critical point is $x=-\frac{1}{2}$.

Second, evaluating f on all critical points and at the endpoints, we get

$$\begin{gathered} f(-1) = \ln(-1)^2+(-1)+1 = \ln 1 = 0, \\ f\left(-\frac{1}{2}\right) = ln\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)+1 = \ln\frac{3}{4}, \text{ and} \\ f(1) = \ln 1^2+1+1 = \ln 3. \end{gathered}$$

And third, we choose our winners. The absolute maximum of $f(x)$ on $[-1,1]$ is $\ln 3$, located at $x=1$. The absolute minimum is $\ln\frac{3}{4}$, located at $x=-\frac{1}{2}$. I will place a graph here once I get my mathematics software functioning again (my computer is getting a mind wipe. So I have only limited software choices.

Exercise 1. Notice by the graph (or by using the Intermediate Value Theorem on the interval $[0,1]$), that there is a point $x=c$, where $f(x)=1$. Find this point.

2. Lecture 2: The Mean Value Theorem

Here is a new question: Suppose $f(x)$ is a continuous function on an interval $[a,b]$ (where $b>a$), and that $f(a)=f(b)$. Must it be the case that there is a critical point in the interval (a,b). See the figure and try to draw a graph of a continuous function that does not have a critical point on (a,b). Think about what a critical point is (the definition is above). If your graph is horizontal for a bit coming out of a, all those points will be critical (0 derivative there). So in your attempt to create a function without critical points, you must either rise out of a, or lower out of it. But then to return to the height of $f(b)$ you will either create a place $c\in(a,b)$ where $f'(c)=0$, or a corner where you stop rising and start lowering. Either of these cases creates a critical point.

Now add the additional criterion that f is not only continuous on $[a, b]$, but differentiable on (a, b). When you have convinced yourself that there must be a critical point in (a, b), the only possibility when f is differentiable on (a, b) is that the critical point $c \in (a, b)$ satisfies $f'(c) = 0$. This is the idea behind yet another existence theorem called ROlle's Theorem:

Theorem 4 (Rolle's Theorem)**.** *Let f be continuous on an interval $[a, b]$, differentiable on (a, b) and satisfies $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Notes:

- Again, there is no information on how to find the point c. But just knowing it exists may be enough for some applications (see below).
- There can be many points of this type (think $g(x) =$ a constant), or just one. But there must be at least one.

Example 5. Show $f(x) = x^3 + x - 1$ has exactly one real root.

Strategy: Again, we cannot "see" the function, so we will rely on the analysis of its properties. For this problem, we will use the Intermediate Value Theorem to show there there is a root. And then we will use Rolle's Theorem to establish that there can be only one. A special note here: This is a problem that ask for an argument more than simply a calculation. This response will look more like a proof than a computation.

Solution: Since $f(x)$ is a polynomial, it is differentiable everywhere (on $(-\infty, \infty)$). Hence it is continuous on any interval. Consider the two values of f: $f(0) = 0^3 + 0 - 1 = -1 < 0$, and $f(1) = 1^3 + 1 - 1 = 1 > 0$. Since f is continuous on $[0, 1]$, and differentiable on $(0, 1)$, we know by the Intermediate Value Theorem, that there exists a point $c \in (0, 1)$ where $f(c) = 0$. Thus $f(x)$ has at least one real root. This root in somewhere between $x = 0$ and $x = 1$.

Now let's suppose there is another real root. Whatever its value as an input, call the lesser of the two roots a and the greater of them b. In this part we will make this assumption of a second root and then see if we can use the assumption to locate the root. Given these two assumed roots, we can immediately say that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), since $f(x)$ is differentiable everywhere, and $f(a) = f(b) = 0$. But then by Rolle's Theorem, we can also say that there must be a point $c \in (a, b)$ where $f'(c) = 0$. But this cannot happen for this function $f(x)$, since $f'(x) = 3x^2 + 1 > 0$ for all values of x. Hence we have arrived at a contradiction. Since our logic was sound, the only possible flaw in the argument was our original assumption, namely that there is a second root of $f(x)$. Hence there is exactly one real root of $f(x)$.

Special Note: This is an example of a "proof by contradiction", a technique of establishing a fact by making an assumption that the fact is false, and then using a series of logical deductions to arrive at a contradiction in the logic. At that point, if the logic is sound, the original assumption is the only possible mistake, and the assumption is thrown out, establishing the fact.

Rolle's Theorem is really only a special case of the following:

Theorem 6 (Mean Value Theorem). *Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b). Then there is a number $c \in (a, b)$, where*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Some notes:

- You can see this IS Rolle's Theorem in the special case that $f(a) = f(b)$.
- The interpretation here is that for a function like the one specified in the theorem, at some point c inside the interval, the instantaneous rate of change of the function with respect to the variable x must be equal to the average rate of change of the function over the entire interval.
- Let $x =$ time for a moment, and the function be distance. Then the interpretation would be that at some point in your trip, as measured by the distance function, your instantaneous velocity (your speedometer reading), must equal the average velocity of your entire trip. For instance, if your average velocity over a trip from Baltimore to New York was 75 miles per hour, (and you operated the car in such a way that your distance varied differentiably with respect to time) there must be a point in your trip when you were traveling at precisely 75 miles per hour. Imagine a state trooper arguing that in court as you fight your speeding ticket!
- Suppose for a moment that the Mean Value Theorem did not specify that the function needs to be differentiable on (a, b). Can the theorem still hold? Convince yourself that the answer is no. As an example, look at the function
$$f(x) = \begin{cases} 2 & 0 \le x \le 7 \\ 4x - 26 & 7 \le x \le 8. \end{cases}$$
Here, the end points of the interval $[0, 8]$ evaluate to $f(0) = 2$ and $f(8) = 6$. The function is continuous on all of $[0, 8]$, and differentiable everywhere on $(0, 8)$ except for $x = 7$. The average rate of change of $f(x)$ on $[0, 8]$ is
$$\frac{f(b) - f(a)}{b - a} = \frac{f(8) - f(0)}{8 - 0} = \frac{6 - 2}{8} = \frac{1}{2}.$$
But the derivative of $f(x)$, where it is defined is $f'(x) = 0$, for $x \in (0, 7)$, and $f'(x) = 4$, for $x \in (7, 8)$. Thus the Mean Value Theorem fails if f is not differentiable on the entire inside of the interval.

We start Section 4.3 with a new question:

Question 7. *What does the derivative of a function say about the behavior of the function?*

This is not an idle question. The derivative is a collection of all of the slopes of the tangent lines to the graph of a function. As linear approximations to the function, these tangent lines must convey important information about how the function behaves across its domain. We begin with a claim:

Claim 1. *If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.*

Intuitively this should make sense. If the slope at a point of the tangent line is positive, then, as the tangent line approximates the function near that point, the function should also "look like" the tangent line, and be increasing there. We can make this more precise, however, now that we have some good machinery like the above theorems.

Proof. Call the interval where $f'(x) > 0$, (a, b), and let x_1 and x_2 be two points inside this interval, where $x_2 > x_1$. Since $f(x)$ is differentiable on (a, b), it is also continuous there. Thus, $f(x)$ will be continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Hence the Mean Value Theorem will hold there, and we can write (using the alternate form in the theorem)

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c \in (x_1, x_2)$. Now we know that $f'(c) > 0$ since the derivative is positive on the entire interval. We also know that $(x_2 - x_1) > 0$ since $x_2 > x_1$. Thus the entire right hand side of the equation is a positive number. Thus we now know that

$$f(x_2) - f(x_1) > 0, \quad \text{or} \quad f(x_2) > f(x_1).$$

As we can choose any two points x_1 and x_2 inside the original interval (a, b), and this will hold, we know that $f(x)$ must be increasing on the entire interval. $\square$

Note that the "sister" claim that $f(x)$ will be decreasing on any interval where $f'(x) < 0$ is proved the same way.

I ended the lecture with a couple of examples.