

110.108 CALCULUS I

Week 4 Lecture Notes: September 19 - September 23

LECTURE 1: SECTION 2.6 LIMITS AT INFINITY

Recall how polynomials behave for very large values of the input variable x (or for very small values (large negative values)). They may bounce around and seemingly oscillate for a while, but at some point they stop this behavior and either tend up or down forever more. One can see this by thinking of the form of a polynomial: It is a sum of multiples of powers of x . The monomial with the largest power will ultimately dominate in determining the value of the function, when the input variable takes large enough values. At that point and forever more, the polynomial will simply either rise or fall.

Keeping in mind that we are studying the “calculus of functions of one independent real variable”, and it is precisely the properties of functions that are the focus of this entire course. One very important property to know is: How does the function behave

- “at it’s tails.”
- “in the long run.”
- “as x goes to infinity (or minus infinity).”

All of these ideas are basically identical. If your function were a function of time, say, and the output variable was the state of a mechanical system, then how the function behaves “in the long run” is a statement about whether the system achieves a steady-state, or equilibrium, or whether it doesn’t. This can be very important information.

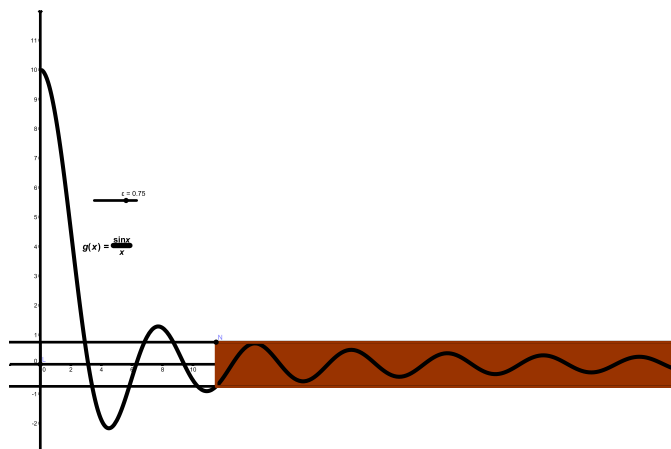
Definition 1. Let f be a function defined on an interval (a, ∞) for a real number a . We say

$$\lim_{n \rightarrow \infty} f(x) = L$$

if, for every real number $\epsilon > 0$, there is an $N > 0$, where

$$\text{If } x > N, \text{ then } |f(x) - L| < \epsilon.$$

This means that if we construct a small horizontal ϵ -band around a number L , for any particular $\epsilon > 0$, then we can find a number N so that for all values of $x > n$, the graph of the function lives *inside* the horizontal band. See the figure.



Some notes:

- There is a similar definition for values of x approaching $-\infty$. This would be the “left tail of the function, as opposed to the above definition which talks about the “right tail”.
- This type of limit is called a *limit at infinity (or minus infinity)*. This new limit is very different from an infinite limit.
- If f has a limit at infinity (or minus infinity), and that limit is L , then we say the line $y = L$ is a *horizontal asymptote* of $f(x)$.
- Contrast this with the following:

Definition 2. Let f be a function where at a real number a , we have one of the four following possibilities:

$$\lim_{x \rightarrow a^\pm} f(x) = \pm\infty.$$

Then the line $x = a$ is called a *vertical asymptote* of $f(x)$.

- It is possible that the limit at infinity is itself infinity (or minus infinity). An example would be the function $f(x) = x$. We have another definition:

Definition 3. $\lim_{x \rightarrow \infty} f(x) = \infty$, if for every positive number M , there is a positive number N where

$$\text{If } x > N, \text{ then } f(x) > M.$$

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Theorem 4. *Polynomials of degree $n > 0$ have no horizontal asymptotes.*

- Section 2.6 has LOTS of examples. You should work through all of them to get these ideas down.
- All limit laws work for limits at infinity.

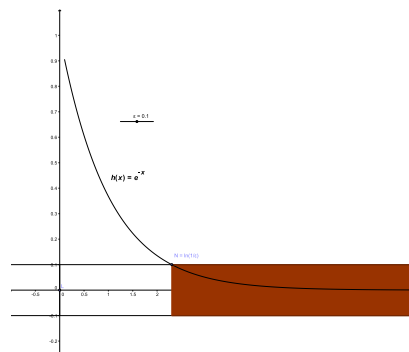
Example 5. For $h(x) = e^{-x}$, show $\lim_{x \rightarrow \infty} h(x) = 0$.

Here, we need to show that given any $\epsilon > 0$, we can find a number N where if $x > N$, then we know

$$|h(x) - 0| = e^{-x} < \epsilon.$$

Notice that $h(x)$, like all exponential functions, is always positive, and hence the absolute values are not needed here.

Let's start with a choice of $\epsilon > 0$. Can we find the N ? First, note that $h(x)$ is a decreasing function. This means that if $x_1 > x_2$, then $h(x_1) < h(x_2)$. This is important, as with the fact that $h(x) > 0$ on its entire domain, it means that if $h(x)$ enters the ϵ -band around $L = 0$, it will never leave it again. Hence to find a good value for N to satisfy the definition, we need only to find a solution to the equation $h(x) = \epsilon$. The solution to this will be our value for N . We can rewrite this as $h(N) = e^{-N} = \epsilon$. This is solved by $N = \ln \frac{1}{\epsilon}$ (Can you find this solution?). As long as $\epsilon > 0$, this N exists. Hence the definition is satisfied and the limit is 0.



Example 6. Show $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$.

Again, you must produce a value of N for any given $\epsilon > 0$. Really, this means that you should find N as a function of ϵ . Follow the pattern of the above example, and you will find that $N = \frac{1}{\epsilon}$ will work here. You should do this in detail.

Two other examples I did:

Example 7. Let $i(x) = \sin x$. Does $\lim_{x \rightarrow \infty} i(x)$ exist?

Here I showed that the limit does not exist at infinity.

Example 8. Let $f(x) = \sqrt{x^2 + 1} - x$. Does $\lim_{x \rightarrow \infty} f(x)$ exist?

Let's do this one in detail. First, given that $f(x)$ is a difference of functions, one can check whether the use of the Difference Limit Law is actually valid here. It would make the calculation easy. We get

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - x \stackrel{?}{=} \lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - \lim_{x \rightarrow \infty} x.$$

Unfortunately, the last term on the right is

$$\lim_{x \rightarrow \infty} x = \infty.$$

As this limit is NOT a real number, the limit does not exist. Thus the Difference Limit Law cannot be used here, and we have to resort to other means.

Next, we try to algebraically rewrite the function in a way which may facilitate our calculations. Whenever one has a difference of two expressions that involve radicals, multiplication by the conjugate (the same expression with the sign changed) may be very helpful. This is because when combining the two binomials, using the FOIL method, the “vowel” terms (in FOIL) cancel out, leaving only the difference in the squared terms. To do this, however, we do not want to change the function, so we multiply with a clever form of 1, using the conjugate. Here

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - x = \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 1} - x \right) \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x} \right) \\ &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 1})^2 - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow \infty} \frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x}. \end{aligned}$$

Arguably, this is an easier limit to calculate since the denominator is never 0 here.

So how do we attack this limit? We can appeal to the definition, and this will work. But there is another way. By using (a slightly modified form of) the Squeezing Theorem. To see this, note the following: For all positive x (we are looking for the limit at $+\infty$), we know $x = \sqrt{x^2} < \sqrt{x^2 + 1}$. But this means that

$$2x = x + x < \sqrt{x^2 + 1} + x, \quad \text{and so } \frac{1}{2x} > \frac{1}{\sqrt{x^2 + 1} + x} \text{ for } x > 0.$$

And as all terms are positive, we then have

$$0 < \frac{1}{\sqrt{x^2 + 1} + x} < \frac{1}{2x}$$

on the interval $(0, \infty)$. And since we now know that

$$\lim_{x \rightarrow \infty} 0 = 0, \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{2x} = \frac{1}{2} \lim_{x \rightarrow \infty} \frac{1}{x} = 0,$$

what can we say about the function in the middle?

Theorem 9. If $f(x) \leq g(x) \leq h(x)$ on an interval (a, ∞) for some real number a , and

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} h(x) = L,$$

then it must be the case that $\lim_{x \rightarrow \infty} g(x) = L$.

This is just another type of squeezing for limits at infinity. In the figure at right, the blue graph is $f(x)$, the function we are calculating the limit at infinity for, and the red graph is of $\frac{1}{2x}$. The green one is the 0-function. Note how close the blue and red graphs get. Our conclusion? By this new form of the Squeezing Theorem, we get $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow \infty} \sqrt{x^2 + 1} - x = 0$.

Again, we first tried to break the function apart and look at the difference of the limits. When this failed (why did it fail?), we multiplied this function by a clever form of 1, using the conjugate of the expression comprising the function. Then

$$f(x) = \frac{1}{\sqrt{x^2 + 1} + x}.$$

It turns out, this is a much easier function to solve. What would the calculation look like if you found this limit by using the definition?

