110.108 CALCULUS I

Week 1 Lecture Notes: September 5 - September 9

Lecture 1: Section 2.1-2.2 Limits

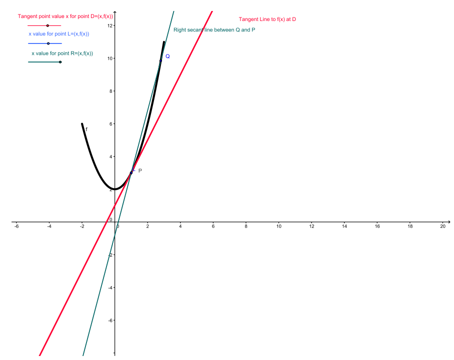
I started with one of the motivational constructions that gives a sense for why the machinery of calculus is necessary to progress past simple mathematical models:

Example 1 (The Tangent Problem.). Construct, the equation of the line tangent to a circle C at a point $p = (x_0, y_0) \in C$. This is actually quite simple and involves only geometrical methods to solve. Note that here it is also quite easy to define what tangency means here, since the tangent line is "the unique line passing through the point which intersects the circle at only one point". The solution? The radial line from the origin to p has slope r. The slope of the line tangent to C at p is perpendicular to this, and hence has slope $m = -\frac{1}{r}$. Hence the equation of the line with slope m passing though p has equation

$$(y - y_0) = m\,(x - x_0)\,.$$

The problem starts when the curve is not a circle. Then we lose the radial line to calculate the tangent line slope. we also lose the idea of what a tangent linea actually is. We can envision it, and can also draw it, if needed. But defining it mathematically is difficult.

As an example, let $f(x) = x^2 + 2$ on the domain $[1, 3]$, and let $P = (x_0, y_0) = (1, 3)$ on the graph (at right). Even though we can envision and draw the tangent line, how can we find its equation, or calculate its slope? The answer is, we fake it by approximating! Take another point on the curve, say $Q = (x_1, y_1) = (x, f(x))$, and calculate the equation of that line passing through Q and P. This may approximate the actual tangent (this line is called a secant line). If we then push Q down to P, we get better and better approximations. The slopes of the resulting secant lines $\overline{QP}$ are all easy to calculate right up to where Q meets P. They are



$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x) - 3}{x - 1}.$$

But when $Q = P$, at the actual point when we need to calculate the slope, The slope equation fails since the change in the x-coordinates (the denominator of the ratio defining the slope) is 0. (Note the change in the y-coordinates is also 0 here, though. This is a very important fact!) Hence our method fails at this point. We need a way to understand that as the second point goes to the first, the resulting lines actually do go to a line, and we should be able to calculate its slope.

This is where the idea of a limit comes from. Basically, it allows us to gather information about how a function behaves at a point, by using only the information about how a function behaves as we approach the point (from each of the sides). Why this is useful is precisely in the instance when the function has a problem at a point, but is perfectly well-behaved around it.

Definition 2 (Intuitive)**.** Suppose $f(x)$ is defined on an open interval point containing a, except possibly at a. Then we write

$$\lim_{x \to a} f(x) = L,$$

and say "the limit of $f(x)$, as x approaches a, is L", if we can make $f(x)$ as close as we like to L by restricting our values of x to be sufficiently close to a (without touching a).

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Some notes here:

- if there is such a number L, it will be a uniquely defined real number. In this case, we say the limit *exists*. If there is no such L, then the limit of $f(x)$ at a does not exist.
- We also write for a limit that exists $f(x) \longrightarrow L$ as $x \longrightarrow a$.
- This must work on both sides of the point a.
- What happens at the point a is of absolutely no importance in this definition.

Example 3 (Jumping frog)**.** A frog jump along lily pads from one side of the pond to the other. His first leap takes exactly half the distance to the other side. As he is now tired, his second jump takes him only half the remaining distance. As he jumps he gets more and more tired, and each time can only make half the previous jump's distance. The big question is: Does he make it to the other side? An equivalent question: Can the sum of the infinite number of positive fractions

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

be 1? Can the sum of an infinite number of positive numbers be finite?! Also, if we let n "go to" infinity, where does the quantity $\frac{2^n-1}{2^n}$ go? This means, does the limit

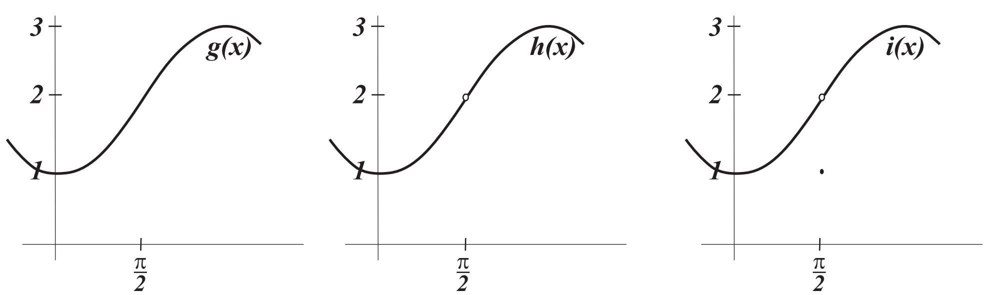
$$\lim_{n\to\infty} \frac{2^n - 1}{2^n}$$

exist? And finally, what happens near 1 here? The answers here may surprise you. But the answers here reveal the essence of some interesting higher math. BTW, the mathematical frog makes it to the other side as long as he is allowed to jump forever.

Example 4. Consider the three functions below. All are slight variations of the function $2 - \cos x$, and are

$$g(x) = 2 - \cos x, \qquad h(x) = 2 - \cos x \text{ on } \left(-\infty, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \infty\right), \qquad i(x) = \begin{cases} 2 - \cos x & x \neq \frac{\pi}{2} \\ 1 & x = \frac{\pi}{2}. \end{cases}$$

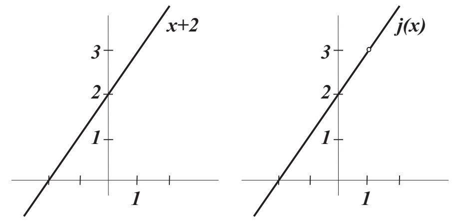
Outside of the point $x = \frac{\pi}{2}$ the functions are all precisely the same. What can you say about the limits of these three functions off the point $x = \frac{\pi}{2}$? What can you say about the limits of these three functions AT the point $x = \frac{\pi}{2}$? What can you say in general about the limits of functions when they agree everywhere but at a few specific isolated points? Remember, in the definition of a limit, what happens at the point a has no influence on what happens around the point a.



Example 5. Let $j(x) = \dfrac{x^2 + x - 2}{x - 1}$. This is a perfectly reasonable rational function. What does the graph look like? It is a line with a hole in it at the point $x = 1$ (why?). Does $\lim\limits_{x\to 1} j(x)$ exist? How can we calculate it if it does?

First, note that $x = 1$ is not in the domain of $j(x)$. But off of this point $j(x)$ is fine and the shape of its graph can be seen by simplifying it:

$$j(x) = \frac{x^2 + x - 2}{x - 1} = \frac{(x+2)\cancel{(x-1)}}{\cancel{x-1}} = x + 2.$$



Hence the graph of $j(x)$ exactly like the graph of the linear function $x+2$ except the the domain does not include $x = 1$ (Note that these functions are NOT the same, but do agree on the domain of $j(x)$). By he previous example, we can see that the limit will exist at $x = 1$, and be the same as that of $x + 2$. Hence

$$\lim_{x \to 1} j(x) = 3.$$

Example 6. Let $k(x) = \frac{(x^2+2)-3}{x-1}$. Does $\lim\limits_{x \to 1} k(x)$ exist? How can we calculate it if it does? Play the same game as you did in the last example, and we see that we can calculate the limit, Indeed,

$$\lim_{x \to 1} k(x) = \lim_{x \to 1} \frac{(x^2 + 2) - 3}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} x + 1 = 2.$$

This calculation rests upon the fact that when off of the point $x = 1$, the factor $(x - 1)$ in both parts of the fraction is a non-zero quantity and can be discarded as a clever form of 1. Thus off of $x = 1$, the function looks exactly like the function and has the same limits.

Note. *Do you recognize the function $k(x)$ in the last example. It is precisely the slope calculation for the function $f(x) = x^2 + 2$ in our earlier example. That was the calculation where we wrote*

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x) - 3}{x - 1} = \frac{(x^2 + 2) - 3}{x - 1}.$$

This calculation broke down at the point we needed to calculate it at (the point P, where $x = 1$). But in this new language, we can use the functions behavior around and near $x = 1$ to conclude that the slope calculation, while not defined at $x = 1$, tends toward a well-defined slope in any case. We will need this later and provides a basic fact in differential calculus. More later.

Lecture 2: Section 2.2-2.3 Limits