CONCEPT: SECTION 4.1 FERMAT'S THEOREM

110.109 CALCULUS I (PHYS SCI & ENG)
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Here is the proof of a theorem, courtesy of Pierre Fermat, which begins the process of locating extrema of continuous functions defined on a closed, bounded interval. The existence of extrema is verified by teh Extreme Value Theorem. The fact that when they exist at a "nice" point (read: one where the derivative exists), the derivative must be 0, means that this is a really good place to hunt for possible extrema: where the derivative is 0:

Theorem 1 (Fermat's Theorem)**.** *If $f(x)$ has a local extremum at a point $x = c$, and $f'(c)$ exists, then it must be the case that $f'(c) = 0$.*

This follows the discussion in class that, for "nice" functions (defined here as those functions that have derivatives), it should be possible to locate some possibilities for extrema by finding places where the tangent line is horizontal (where the derivative is 0, or as we say, where the derivative *vanishes*). It turns out that by this theorem, this is indeed the case.

Proof. Suppose for now that f has a local maximum at $x = c$. By definition, this means that for all points x near c (read: in a small open interval containing c), we know that $f(x) \leq f(c)$. To make this a bit more precise, choose a small positive $h > 0$, small enough that for all $x \in [c, c+h)$, $f(c+h) \leq f(c)$. We are only working to the right of c at the moment, and since $f(c)$ is a local max, this is possible. Now since $f(c+h) \leq f(c)$, we know also that

$$f(c+h) - f(c) \leq 0, \quad \text{and} \quad \frac{f(c+h) - f(c)}{h} \leq \frac{0}{h} = 0,$$

since h is positive (the sense does not change in the equality by division with a positive number). For all different values of $h > 0$ this remains true, and hence by Theorem 2.3.2 in Chapter 2 (thinking of each side as a function of h), we also know that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} = 0.$$

But the left hand side of the inequality is precisely the right-side limit for the definition of $f'(c)$, which we know exists by supposition (and hence must equal its side limits!). Hence this last statement says that the derivative of f at $x = c$ cannot be positive, and

$$f'(c) \leq 0.$$

Now let $h < 0$ be a small negative number (and hence we will work on the left side of $x = c$. Again, since $f(c)$ is a local maximum, if we choose h small enough, we know that $f(c+h) \leq f(c)$.

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Note that this looks exactly the same, yet it is not since we have $h < 0$ here. Why this is different is that here, we see that

$$f(c+h) - f(c) \leq 0, \quad \text{while} \quad \frac{f(c+h)-f(c)}{h} \geq \frac{0}{h} = 0.$$

Why this is true should be elementary: When we divide by a small negative number h, we change the sense of the inequality, right? Following above (and reusing Theorem 2.3.2), we then see that now

$$\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq \lim_{h \to 0^-} = 0.$$

The left hand side of the inequality is now the left-side limit of $f'(c)$ (which exists and must equal both its side limits). But this last statement says that the derivative of f at $x = c$ cannot be negative, and

$$f'(c) \geq 0.$$

For $f'(c)$ to satisfy both of these inequalities, it can only be the case that $f'(c) = 0$. $\square$

A final note: If you think of what the graph of a local maximum looks like, the whole notion of the derivative (as the slope of the tangent line) as a limit of the slopes of secant line along the curve should make sense here. On the right, the function should dip down (or stay the same height. The the slopes of ALL of the secant lines should dip down (or stay the same height). Thus the limit should either dip down or stay the same height (slope 0). On the left, the slopes of ALL of the secant lines should either be positive or 0, hence this should also be the case in the limit. Think about this and draw pictures. You will see it soon enough (if you do not already).