EXAMPLE: IMPLICIT DIFFERENTIATION

110.109 CALCULUS I (PHYS SCI & ENG)
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Question 1. *Given* $\frac{1}{x} + \frac{1}{y} = 2xy$, *do the following:.*

(a) Calculate $\left.\frac{dy}{dx}\right|_{(1,1)}$.

(b) Find all places where the graph of the equation has a horizontal tangent line (where $\frac{dy}{dx} = 0$).

Part (a) Strategy: Since we cannot solve this equation for either variable as a function of the other, we will instead differentiate the entire equation to find the derivative of y with respect to x.

Part (a) Solution: First, note that the point $(x, y) = (1, 1)$ does satisfy the equation and so is on the graph of solutions. As for the derivative, we have

$$\frac{d}{dx}\left[\frac{1}{x} + \frac{1}{y} = 2xy\right] \quad \Longrightarrow \quad -\frac{1}{x^2} - \frac{1}{y^2} \cdot \frac{dy}{dx} = 2(1)y + 2x\left(\frac{dy}{dx}\right) = 2y + 2x\frac{dy}{dx}.$$

Note we needed the Product Rule on the right-hand side. We can now solve for the derivative (as we ALWAYS can in these situations):

$$\frac{dy}{dx} = \frac{-\frac{1}{x^2} - 2y}{\frac{1}{y^2} + 2x}. \tag{1}$$

There is no need to simplify here. At the point $x = 1$ and $y = 1$, we get

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-\frac{1}{(1)^2} - 2(1)}{\frac{1}{(1)^2} + 2(1)} = -\frac{3}{3} = -1.$$

Hence $\left.\frac{dy}{dx}\right|_{(1,1)} = -1$.

Part (b) Strategy: Here, we use the general form for the derivative (Equation 1 found in Part (a)), and set it to 0. This will give us another equation involving x and y. We will use this new equation, along with the original one, to find the points which have derivative 0 and are simultaneously on the graph of the original equation.

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Part (b) Solution: First, the only solutions to

$$\frac{dy}{dx} = \frac{-\frac{1}{x^2} - 2y}{\frac{1}{y^2} + 2x}$$

are where $-\dfrac{1}{x^2} - 2y = 0$. This is because the function is a compound fraction. Along the curve in the plane given by $y = -\frac{1}{2x^2}$, the numerator is 0 while the denominator is not. However, not all points in the plane that satisfy this equation actually lie on the graph of the original curve. To finds those, we wind up solving 2 equations in two unknowns:

$$\frac{1}{x} + \frac{1}{y} = 2xy$$
$$-\frac{1}{x^2} - 2y = 0.$$

To do this, solve one equation for one of the variables, and substitute it into the other. Then you wind up with one equation in one unknown, which is usually easier to solve. We take the second, as $y = -\frac{1}{2x^2}$, and substitute into the first to get

$$\frac{1}{x} + \frac{1}{-\frac{1}{x^2}} = 2x\left(-\frac{1}{x^2}\right), \quad \Longrightarrow \quad \frac{1}{x} - 2x^2 = -\frac{2}{x}.$$

We get

$$1 - 2x^3 = -1, \quad \Longrightarrow \quad 2x^3 = 2, \quad \Longrightarrow \quad x^3 = 1,$$

solved only by $x = 1$.

Note: The point in Part (a), $(x, y) = (1, 1)$, had x-coordinate 1, and yet the derivative there was -1. What gives? Only that if there are any points on the curve that have a 0 derivative, their x-coordinate would have to be 1. It does not mean that all points with x-coordinate 1 have to have a 0 derivative!

Back to the problem: We still need to find the y-coordinates of the points with 0 derivative. We can use either equation in the pair above to do this. The easiest is the 0-derivative equation $y = -\frac{1}{2x^2}$. Here

$$y = -\frac{1}{2(1)^2} = -\frac{1}{2},$$

and this is the ONLY one. The other point form Part(a) does not satisfy the derivative equation.

The ONLY point with a 0-derivative is the point $(x, y) = \left(1, -\dfrac{1}{2}\right)$. The graph of solutions to this original equation is not easy to see. But look at the figure to see that we are correct here. There are two point on the graph with x-coordinate 1. The one in Part (a) is at $(1, 1)$. The other below it is the one here in Part (b).