CONCEPT: SECTION 3.3 THE DERIVATIVE OF THE SINE FUNCTION

110.109 CALCULUS I (PHYS SCI & ENG)
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Theorem 1. *Show* $\frac{d}{dx}[\sin x] = \cos x$.

Note: One can plotting a few values of the slopes of lines tangent to the function $f(x) = \sin x$ to see that this is true. One can also plot both $f(x)$ and $f'(x) = \cos x$, one over the other, to match up the values of tangent line slopes to function values. This is the gist of the Geogebra applet I placed on the website homepage. But to actually "see" that the derivative of sine is cosine, one needs a bit of analysis. Here, we write out that proof, as done in class. I hope you find this helpful.

First, appealing to the definition of the derivative, we see that

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}.$$

There are many very useful identities one can use when working with trigonometric functions. One that works here is:

$$\sin(a+b) = \sin a \cos b + \sin b \cos a.$$

Using this, we get

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \\
&= \left(\lim_{h\to 0} \sin x\right)\left(\lim_{h\to 0} \frac{(\cos h - 1)}{h}\right) + \left(\lim_{h\to 0} \frac{\sin h}{h}\right)\left(\lim_{h\to 0} \cos x\right),
\end{aligned}$$

where, of course, breaking up a limit in to a product and sum of limits only works when the individual limits exist. As long as they ultimately do, we will be fine. Note that two of these limits do not directly involve the variable h. Hence these limits are readily evaluated, exist and are equal

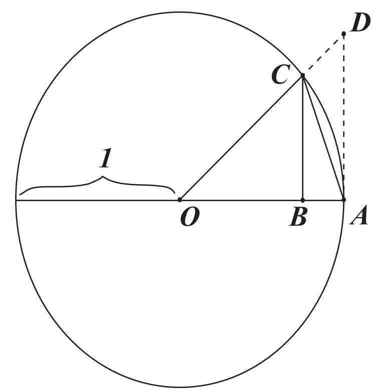
Date: September 28, 2011.

to the expression (they are a constant to the variable h). Hence up to now, we know that

$$\frac{d}{dx}\left[\sin x\right] = \sin x \left(\lim_{h\to 0} \frac{(\cos h - 1)}{h}\right) + \left(\lim_{h\to 0} \frac{\sin h}{h}\right)\cos x.$$

Already, we now know that the derivative of the sine function is some combination of the sine function and the cosine function, as long as the two remaining limits exist. Let's evaluate them now:

Claim 1. $\displaystyle\lim_{h\to 0} \frac{\sin h}{h} = 1.$



To see this, we will need some geometry: In the figure at right is the unit circle (the circle centered at the origin with radius 1), with some points and geometric figures drawn in. Note that the coordinates of point C in the figure are $C = (\cos x, \sin x)$. Why is this? Look at the right triangle given by the three points O, B, and C. The hypotenuse of this triangle is 1. Hence,

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenus}} = \text{length of}\overline{BC}.$$

Similarly, the length of $\overline{OB}$ is $\cos\theta$, and the length of $\overline{AD}$ is $\tan\theta$ (you should convince yourself of this last one on your own!). Now, given the diagram, note the relationships:

$$\begin{array}{ccccc} (\text{Area of triangle } AOD) & > & (\text{Area of sector } AOC) & > & (\text{Area of triangle } AOC) \\ \frac{1}{2}(1)\frac{\sin\theta}{\cos\theta} & > & \frac{1}{2}(1)^2\theta & > & \frac{1}{2}(1)\sin\theta. \end{array}$$

We can clean this up by multiplying all three parts of this inequality by the factor $\frac{2}{\sin\theta}$, noting that this will not change the sense of the inequality (Why is this?). We get

$$\frac{1}{\cos\theta} \quad > \quad \frac{\theta}{\sin\theta} \quad > \quad 1.$$

And since all of these are positive when $\theta > 0$, we can also invert the parts of this inequality, which does reverse the sense (read direction) of the inequality:

$$\cos\theta \quad < \quad \frac{\sin\theta}{\theta} \quad < \quad 1.$$

What we have provided here is a means of evaluating the limit we have claimed. Indeed, in this last inequality, we see that our function in the middle is sandwiched between two other functions, both of which are continuous at and near 0. And both satisfy

$$\lim_{\theta\to 0} \cos\theta = 1 = \lim_{\theta\to 0} 1.$$

Hence by the Squeezing Theorem, we see that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

also. But this is the claim.

Claim 2. $\lim_{\theta \to 0} \frac{\cos h - 1}{h} = 0.$

This one is a little more conventional to evaluate. One must think of conjugates again here (and another clever form of 1) to see what is happening. Think that the expression in the limit is not continuous at $h = 0$, hence one cannot simply "plug $h = 0$ into the expression". However, since both the numerator and denominator go to 0 as h goes to 0, there is hope that this limit exists. We try some algebraic manipulation to re-arrange the expression without changing it. Indeed,

$$\lim_{\theta \to 0} \frac{\cos h - 1}{h} = \lim_{\theta \to 0} \frac{\cos h - 1}{h} \left(\frac{\cos h + 1}{\cos h + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2 h - 1}{h \left(\cos h + 1\right)}.$$

While this may not look so helpful, it turns out to be very helpful. We still cannot evaluate the limit directly by limit laws, since the denominator and numerator still both go to 0. But with a little more manipulation (and the use of the identity $1 = \sin^2 h + \cos^2 h$), we can pull this limit apart to get

$$\lim_{\theta \to 0} \frac{\cos^2 h - 1}{h \left(\cos h + 1\right)} = \lim_{\theta \to 0} \frac{-\sin^2 h}{h \left(\cos h + 1\right)} = \left(\lim_{\theta \to 0} \frac{\sin h}{h} \right) \left(\lim_{\theta \to 0} \frac{-\sin h}{\cos h + 1} \right).$$

The first factor is what we just calculated in Claim 1, and evaluates to 1. The second limit has an expression that is actually continuous at $h = 0$ (the numerator and denominator both individually have limits and the denominator limit is not 0. Hence the quotient Limit Law works here. We get

$$\lim_{\theta \to 0} \frac{-\sin h}{\cos h + 1} = \frac{0}{1+1} = 0.$$

This means that the claim is established.

Back to our original calculations, and using these last two claims, we get

$$\frac{d}{dx}\left[\sin x\right] = \sin x \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \cos x. = (\sin x)(0) + (1) \cos x = \cos x.$$