CONCEPT: SECTION 2.8 THE DERIVATIVE OF A FUNCTION

110.109 CALCULUS I (PHYS SCI & ENG)
PROFESSOR RICHARD BROWN

Theorem 1. *If $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at a.*

Note: Since the definition of a derivative utilizes a limit, it is not obvious at first that this needs to be true, since we do not care what happens at a point a to discuss a limit as x approaches a. However, the derivative is a limit of a ratio of differences between function values over differences in input values and not simply a limit of a function. It turns out that the three things that determine continuity must be present: (1) a must be in the domain of f, (2) all values sufficiently close to a must also be in the domain of f, and (3) the numerator of the derivative definition must go to 0 as the denominator goes to 0. Else the limit would not exist. But this last part means precisely that the definition of continuity must hold. Here is the proof:

Proof. Given that the derivative of $f(x)$ exist at a, we know that

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. It is certainly true that this ratio of differences exists when x is only close to and not equal to a. We can abuse this to write

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a),$$

noting that this is obviously true, no? But then at the point $x = a$, we can write

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) = \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \lim_{x \to a}(x - a)$$

by the Product Limit Law (as long as the two limits on the far right exist, that is). But they do, and we can see

$$\left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0.$$

Hence $\lim_{x \to a} f(x) - f(a) = 0$. But this is really exactly what continuity is at z. To see this, let's calculate the formula for continuity directly:

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(x) + (-f(a) + f(a)).$$

Date: September 26, 2011.

This is one of our "clever forms of 0" that I talked about in class. Basically, it does not change an expression's values, but it does change its form, and if done properly can expose structure that was hidden before. Here we continue, and note that, via the Sum Limit Law

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} f(x) + (-f(a) + f(a)) \\
&= \lim_{x\to a} (f(x) - f(a)) + f(a) \\
&= \lim_{x\to a} (f(x) - f(a)) + \lim_{x\to a} f(a) \\
&= 0 + f(a) = f(a).
\end{aligned}$$

Hence, $f(x)$ is continuous at $x = a$ and we are done. □