110.107 - Homework #9 Solutions

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Section #1

Section 10.5

2. Let $f(x,y) = e^x \sin y$ with $x(t) = t$ and $y(t) = t^3$. Find the derivative of $w(x,y)$ with respect to t at $t=1$.

Answer: The chain rule states that $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$= (e^x \sin y)(1) + (e^x \cos y)(3t^2) = e^t \sin(t^3) + 3t^2 e^t \cos(t^3)$.

When $t=1$, $\left.\frac{dw}{dt}\right|_{t=1} = e^1 \sin 1 + 3e^1 \cos 1 = \boxed{e(\sin 1 + 3\cos 1)}$.

12. Find $\frac{dy}{dx}$ if $\cos(x^2+y^2) = \sin(x^2-y^2)$.

Answer: From page 539, we have that if $w = F(x,y)$ is differentiable and $F(x,y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad \text{when } F_y \neq 0.$$

Here, let $F(x,y) = \sin(x^2-y^2) - \cos(x^2+y^2)$, and set $F(x,y) = 0$.

Then

$$F_x = \frac{\partial}{\partial x}F(x,y) = 2x\cos(x^2-y^2) + 2x\sin(x^2+y^2).$$

$$F_y = \frac{\partial}{\partial y}F(x,y) = -2y\cos(x^2-y^2) + 2y\sin(x^2+y^2).$$

Thus

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{x}{y}\left(\frac{\cos(x^2-y^2)+\sin(x^2+y^2)}{-\cos(x^2-y^2)+\sin(x^2+y^2)}\right)}.$$

16. Suppose that you travel along an environmental gradient, along which both temperature and precipitation increase. If the abundance of a particular plant species increases with both temperature and precipitation, would you expect to encounter this species more often or less often during your journey? (Use calculus to answer this question)

Answer: Okay, so we've got a function for the population of some plant species in terms of temperature and precipitation. Let f be the population, T the temperature, and P the precipitation. So $f = f(P,T)$. Our path is a curve $\vec{r}(t) = \begin{bmatrix} P(t) \\ T(t) \end{bmatrix}$, where t is time. Also, our path is along the gradient of f, thus it is parallel to the gradient of f at t for all t. So we have $\vec{r}\,'(t) = \lambda \nabla f$ for some positive $\lambda \in \mathbb{R}$.

By the chain rule $\frac{df}{dt} = \frac{\partial f}{\partial P}\frac{dP}{dt} + \frac{\partial f}{\partial T}\frac{dT}{dt} = \nabla f \cdot \vec{r}\,'(t) = \nabla f \cdot (\lambda \nabla f)$

$$= \lambda |\nabla f|^2 > 0,$$

since $\lambda > 0$ and $|\nabla f|^2 > 0$. Thus as time increases - and you travel along the path - you will see an increase in the plant species's population.

18. Find ∇f for $f(x,y) = \frac{xy}{x^2+y^2}$.

Answer: $\nabla f = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] = \left[\frac{\partial}{\partial x}\left[\frac{xy}{x^2+y^2}\right], \frac{\partial}{\partial y}\left[\frac{xy}{x^2+y^2}\right]\right]$

$$= \left[\frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2}, \frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2}\right]$$

$$\boxed{= \left[\frac{y^3 - x^2y}{(x^2+y^2)^2}, \frac{x^3 - y^2x}{(x^2+y^2)^2}\right]}$$

20. Find ∇f for $f(x,y) = x(x^2-y^2)^{2/3}$.

Answer: $\nabla f = \left[\frac{\partial}{\partial x}\left[x(x^2-y^2)^{2/3}\right], \frac{\partial}{\partial y}\left[x(x^2-y^2)^{2/3}\right]\right]$

$$= \left[\frac{2}{3}(x^2-y^2)^{-1/3}(x)(2x) + (x^2-y^2)^{2/3}, \frac{2}{3}(x^2-y^2)^{-1/3}(-2y)(x)\right]$$

$$\boxed{= \left[\frac{4}{3}\frac{x^2}{(x^2-y^2)^{1/3}} + (x^2-y^2)^{2/3}, -\frac{4}{3}xy(x^2-y^2)^{-1/3}\right]}.$$

28. Find the directional derivative of $f(x,y) = x^3y^2$ at $(2,3)$ in the direction $\begin{bmatrix}-2\\1\end{bmatrix}$.

Answer: The directional derivative of a function $f(x,y)$ at (x_0, y_0) in the direction of the vector $\vec{v}$ is

$$D_{\vec{v}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \frac{\vec{v}}{|\vec{v}|}.$$

So $\nabla f = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix} = \begin{bmatrix}3x^2y^2 & 2x^3y\end{bmatrix}$,

$\nabla f(2,3) = \begin{bmatrix}3\cdot 2^2\cdot 3^2 & 2\cdot 2^3\cdot 3\end{bmatrix} = \begin{bmatrix}108 & 48\end{bmatrix}$, and hence

$$D_{[-2\ 1]} f(2,3) = \begin{bmatrix}108 & 48\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}\frac{1}{\sqrt{(-2)^2+1^2}} = \begin{bmatrix}-216+48\end{bmatrix}\frac{1}{\sqrt{5}} = \boxed{-\frac{168}{\sqrt{5}}}.$$

30. Find the directional derivative of $f(x,y) = ye^{x^2}$ at $(0,2)$ in the direction $\begin{bmatrix}4\\-1\end{bmatrix}$.

Answer: $\nabla f = \begin{bmatrix}2xye^{x^2} & e^{x^2}\end{bmatrix}$; $\nabla f(0,2) = \begin{bmatrix}2(0)(2)e^{0^2} & e^{0^2}\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}$.

$$D_{[4\ -1]} f(0,2) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix}\frac{1}{\sqrt{4^2+(-1)^2}} = \boxed{-\frac{1}{\sqrt{17}}}.$$

32. Find the directional derivative of $f(x,y) = 4xy + y^2$ at the point $P = (-1,1)$ in the direction of $Q = (3,2)$.

Answer: Our direction vector is $\vec{Q} - \vec{P} = \begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix}$.

$\nabla f = \begin{bmatrix}4y & 4x+2y\end{bmatrix}$; $\nabla f(-1,1) = \begin{bmatrix}4(1) & 4(-1)+2(1)\end{bmatrix} = \begin{bmatrix}4 & -2\end{bmatrix}$.

So $D_{[4\ 1]} f(-1,1) = \begin{bmatrix}4 & -2\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\frac{1}{\sqrt{4^2+1^2}} = (16-2)\frac{1}{\sqrt{17}} = \boxed{\frac{14}{\sqrt{17}}}$.

36. In what direction does $f(x,y) = \sqrt{x^2-y^2}$ increase most rapidly at $(5,3)$?

Answer: If f is a differentiable function, then f increases most rapidly in the direction $\nabla f(x_0,y_0)$ at each point (x_0,y_0).

We have that $\nabla f = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{x}{\sqrt{x^2-y^2}}, & \frac{-y}{\sqrt{x^2-y^2}}\end{bmatrix}$, and

$\nabla f(5,3) = \begin{bmatrix}\frac{5}{\sqrt{25-9}}, & \frac{-3}{\sqrt{25-9}}\end{bmatrix} = \begin{bmatrix}\frac{5}{4} & -\frac{3}{4}\end{bmatrix}$. So f increases most rapidly in the direction $\begin{bmatrix}5/4\\-3/4\end{bmatrix}$ at $(5,3)$.

40. Find a unit vector that is normal to the level curve of the function $f(x,y) = x^2 + \frac{y^2}{9}$ at the point $(1,3)$.

Answer: The gradient of a function $f(x,y)$ is always perpendicular to the level curve $f(x,y) = c$ at any (x_0, y_0) on the level curve. So we have

$$\nabla f = \begin{bmatrix} 2x & \frac{2y}{9} \end{bmatrix} ; \quad \nabla f(1,3) = \begin{bmatrix} 2 & \frac{6}{9} \end{bmatrix} = \begin{bmatrix} 2 & \frac{2}{3} \end{bmatrix}. \text{ So}$$

$\begin{bmatrix} 2 \\ 2/3 \end{bmatrix}$ is perpendicular to the level curve of f at $(1,3)$.

The unit vector is $\begin{bmatrix} 2 \\ 2/3 \end{bmatrix} \frac{1}{\sqrt{2^2 + (2/3)^2}} = \begin{bmatrix} 2 \\ 2/3 \end{bmatrix} \frac{1}{\sqrt{4 + \frac{4}{9}}}$

$$= \begin{bmatrix} 2 \\ 2/3 \end{bmatrix} \frac{3}{2\sqrt{10}} = \boxed{\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}.$$

44. Suppose an organism moves down a sloped surface along the steepest line of descent. If the surface is given by $f(x,y) = x^2 - y^2$, find the direction in which the organism will move at the point $(2,3)$.

Answer: Let $\vec{u}$ be a unit vector. Then the directional derivative at $(2,3)$ in the direction of $\vec{u}$ is

$$(D_{\vec{u}} f)(2,3) = (\nabla f)(2,3) \cdot \vec{u} = |(\nabla f)(2,3)| |\vec{u}| \cos\theta = |(\nabla f)(2,3)| \cos\theta,$$

where θ is the angle between $(\nabla f)(2,3)$ and $\vec{u}$. The direction of steepest descent occurs when $(D_{\vec{u}} f)(2,3)$ is most negative, i.e. when $\cos\theta = -1$, or $\theta = \pi$. Thus the direction of steepest descent is $-(\nabla f)(2,3)$:

$$\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix} ; \quad -(\nabla f)(2,3) = -\begin{bmatrix} 2(2) \\ -2(3) \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ 6 \end{bmatrix}}.$$

Section 10.5

In problems 2, 4, 6, 8, and 10, the functions are defined for all $(x,y) \in \mathbb{R}^2$. Find all candidates for local extrema, and use the Hessian matrix to determine their types.

2. $f(x,y) = -2x^2 - y^2 + 3y$

<u>Answer</u>: Since f is differentiable, its critical points occur where $\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -4x \\ -2y+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Clearly the only critical point is $(0, \frac{3}{2})$.

The Hessian matrix is

$$\text{Hess} f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix}.$$

To determine the nature of our critical point, use the test on p. 553:

SECOND DERIVATIVE TEST FOR FUNCTIONS OF TWO VARIABLES (TO $\mathbb{R}$)

Suppose the second partial derivatives of $f: \mathbb{R}^2 \to \mathbb{R}$ are continuous in a disk centered at (x_0, y_0), and $\nabla f(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then

1. If $\det(\text{Hess} f) > 0$ and $\text{tr}(\text{Hess} f) > 0$ at (x_0, y_0), then f has a local minimum at (x_0, y_0).
2. If $\det(\text{Hess} f) > 0$ and $\text{tr}(\text{Hess} f) < 0$ at (x_0, y_0), then f has a local maximum at (x_0, y_0).
3. If $\det(\text{Hess} f) < 0$, then f has a saddle point at (x_0, y_0).

There is one case left, not listed in the book:

4. If $\det(\text{Hess} f) = 0$, then this test is inconclusive.

Now, 4 does NOT mean it's impossible to know whether the critical point is a local maximum, minimum, or saddle point if $\det(\text{Hess} f) = 0$ at (x_0, y_0); it just means you have to use some other method to figure out what the critical point is.

~~Th~~ Try to show that $f(x,y) = x^4 + y^4$, $g(x,y) = -x^4 - y^4$, and $h(x,y) = x^4 - y^4$ all have critical points at $(0,0)$, all have $\det(\text{Hess} \,\cdot\,) = 0$ at $(0,0)$, and ~~[illegible]~~ f has a local minimum at $(0,0)$, g a local maximum, and h a saddle point.

Anyway, let's use the test on this problem:

$$\det(\text{Hess } f) = \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = (-4)(-2) - (0)(0) = 8$$

$$\text{tr}(\text{Hess } f) = -4 - 2 = -6.$$

So the test says that $\boxed{f(x,y) = -2x^2 - y^2 + 3y \text{ has a local maximum at } (0, \tfrac{3}{2}).}$

4. $f(x,y) = xy - 2y^2$.

Answer: Critical points occur where $\nabla f = 0$:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x - 4y. \text{ So } \begin{cases} y = 0 \\ x - 4y = 0 \end{cases} \longrightarrow x = 0.$$

The only critical point is at $(0,0)$.

$$\text{Hess } f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}. \text{ So } \begin{cases} \det(\text{Hess } f) = -1 \\ \text{tr}(\text{Hess } f) = -4 \end{cases}.$$

$\boxed{\text{Thus } (0,0) \text{ is a saddle point.}}$

6. $f(x,y) = x(1 - x + y)$.

Answer: First, write $f(x,y) = x - x^2 + xy$. Critical points occur where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$:

$$\frac{\partial f}{\partial x} = 1 - 2x + y \qquad \frac{\partial f}{\partial y} = x. \text{ So } \begin{cases} x = 0 \\ 1 - 2x + y = 0 \end{cases} \rightarrow y = -1.$$

The only critical point is at $(0,-1)$.

$$\text{Hess } f = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}. \text{ So } \begin{cases} \det(\text{Hess } f) = -1 \\ \text{tr}(\text{Hess } f) = -2 \end{cases}.$$

$\boxed{\text{Thus } (0,-1) \text{ is a saddle point.}}$

8. $f(x,y) = yxe^{-y}$.

Answer: Critical points occur when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$:

$$\frac{\partial f}{\partial x} = ye^{-y} \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[yxe^{-y}] = yx\frac{\partial}{\partial y}[e^{-y}] + e^{-y}\frac{\partial}{\partial y}[yx]$$
$$= -yxe^{-y} + xe^{-y}.$$

So we have the system

$$\begin{cases} ye^{-y} = 0 \\ -yxe^{-y} + xe^{-y} = 0 \end{cases}.$$

Note that e^{-y} cannot equal 0 (it is positive for all y); so we can divide both sides by e^{-y} to get

$$\begin{cases} y = 0 \\ -yx + x = 0 \end{cases}.$$

So the only critical point is $(0,0)$.

$$\text{Hess } f = \begin{bmatrix} 0 & -ye^{-y} + e^{-y} \\ -ye^{-y} + e^{-y} & +yxe^{-y} - xe^{-y} - xe^{-y} \end{bmatrix} = \begin{bmatrix} 0 & -ye^{-y} + e^{-y} \\ -ye^{-y} + e^{-y} & yxe^{-y} - 2xe^{-y} \end{bmatrix}.$$

So

$$(\text{Hess } f)(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{matrix} \det[(\text{Hess} f)(0,0)] = -1 \\ \operatorname{tr}[(\text{Hess} f)(0,0)] = 0 \end{matrix}.$$

$\boxed{\text{Thus } (0,0) \text{ is a saddle point.}}$

10. $f(x,y) = y \sin x$.

Answer: $\frac{\partial f}{\partial x} = y\cos x \qquad \frac{\partial f}{\partial y} = \sin x$. So we have $\begin{cases} y\cos x = 0 \\ \sin x = 0 \end{cases}$.

$\sin x = 0$ gives $x = n\pi$ for some integer n. If $y\cos x = 0$, then either $y = 0$ or $\cos x = 0$. But as $x = n\pi$, $\cos x = \cos(n\pi) = (-1)^n \neq 0$. So $y = 0$ and $x = n\pi$. Thus there is a critical point at $(n\pi, 0)$ for all $n \in \mathbb{Z}$.

$$\text{Hess } f = \begin{bmatrix} -y\sin x & \cos x \\ \cos x & 0 \end{bmatrix}. \text{ So } \det(\text{Hess } f) = -\cos^2 x < 0 \text{ for all } x.$$

So $\boxed{f \text{ has saddle points at } (n\pi, 0) \text{ for all } n \in \mathbb{Z}.}$

12. Consider the function $f(x,y) = ax^2 + by^2$.

(a) Show that $(\nabla f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) Find values for a and b such that $(0,0)$ is (i) a local minimum, (ii) a local maximum, and (iii) a saddle point.

<u>Answer</u>: (a) $\nabla f = \begin{bmatrix} 2ax \\ 2by \end{bmatrix}$, so $(\nabla f)(0,0) = \begin{bmatrix} 2a(0) \\ 2b(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) $\text{Hess} f = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$, so $\det(\text{Hess} f) = 4ab$, $tr(\text{Hess} f) = 2(a+b)$.

(i) $(0,0)$ is a local minimum if $\det(\text{Hess } f) > 0$ and $tr(\text{Hess } f) > 0$.

Thus $4ab > 0$ and $2(a+b) > 0$. [redacted]. I was going to solve for the possible values of a and b, but that would take too long. Just pick a and b such that $4ab > 0$ and $2(a+b) > 0$. $\boxed{a = b = 1 \text{ works.}}$

(ii) $(0,0)$ is a local maximum if $\det(\text{Hess } f) > 0$ and $tr(\text{Hess } f) < 0$.

Thus $4ab > 0$ and $2(a+b) < 0$. $\boxed{a = b = -1 \text{ works.}}$

(iii) $(0,0)$ is a saddle point if $4ab < 0$. $\boxed{\text{So } a = 1, b = -1 \text{ works.}}$