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Section #1

Section 10.4

In problems 2, 4, and 6, find the tangent plane to the given function at the given point.

2. $f(x,y) = x^2 - 3y^2$; $(-1,1,-2)$

Answer: If the tangent plane to a function $f(x,y)$ exists at a point $P = (x_0, y_0, z_0)$, then its equation is

$$L(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(x_0,y_0)} (y - y_0).$$

(note that $z_0 = f(x_0,y_0)$). Here,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2 - 3y^2] = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2 - 3y^2] = -6y,$$

so

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)=(-1,1)} = 2(-1) = -2 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(-1,1)} = -6(1) = -6.$$

Finally, $f(-1,1) = (-1)^2 - 3(1)^2 = 1 - 3 = -2$ (as you should have guessed), so the equation for the tangent plane is

$$L(x,y) = -2(x - (-1)) - 6(y-1) - 2$$
$$= -2x - 2 - 6y + 6 - 2 = -2x - 6y + 2.$$

The answer is $\boxed{L(x,y) = -2x - 6y + 2.}$

4. $f(x,y) = \sin x + \cos y$; $(0,0,1)$.

<u>Answer</u>: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin x + \cos y] = \cos x$; $\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \cos 0 = 1$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin x + \cos y] = -\sin y$; $\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = -\sin 0 = 0$

So the tangent plane is

$$L(x,y) = 1(x-0) + 0(y-0) + 1$$

$$\boxed{L(x,y) = x+1.}$$

6. $f(x,y) = e^{x-y}$; $(1,-1,e^2)$

<u>Answer</u>: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[e^{x-y}] = e^{x-y}\frac{\partial}{\partial x}[x-y] = e^{x-y}$

$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,-1)} = e^{1-(-1)} = e^2$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[e^{x-y}] = e^{x-y}\frac{\partial}{\partial y}[x-y] = -e^{x-y}$

$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,-1)} = -e^{1-(-1)} = -e^2$.

So the tangent plane is

$$L(x,y) = e^2(x-1) - e^2(y+1) + e^2$$

$$\boxed{L(x,y) = e^2(x-y-1).}$$

14. Show that $f(x,y) = e^{x-y}$ is differentiable at $(0,0)$.

<u>Answer</u>: We have that $\frac{\partial f}{\partial x} = e^{x-y}$ and $\frac{\partial f}{\partial y} = -e^{x-y}$. Since f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are continuous everywhere, f is differentiable on all of $\mathbb{R}$.
(See the "Sufficient Condition for Differentiability" on p. 530.)

20. Find the linearization of $f(x,y) = \cos(x^2 y)$ at $(\pi/2, 0)$.

Answer: The linearization is just the tangent plane:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\cos(x^2 y)] = -\sin(x^2 y)\frac{\partial}{\partial x}[x^2 y] = -2xy\sin(x^2 y)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(\frac{\pi}{2},0)} = -2\left(\frac{\pi}{2}\right)(0)\sin\left(\left(\frac{\pi}{2}\right)^2(0)\right) = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\cos(x^2 y)] = -\sin(x^2 y)\frac{\partial}{\partial y}[x^2 y] = -x^2\sin(x^2 y)$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(\frac{\pi}{2},0)} = -\left(\frac{\pi}{2}\right)^2\sin\left[\left(\frac{\pi}{2}\right)^2 0\right] = 0$$

$$f\left(\frac{\pi}{2},0\right) = \cos\left[\left(\frac{\pi}{2}\right)^2 0\right] = \cos 0 = 1.$$

So the linearization is

$$L(x,y) = 0\left(x-\frac{\pi}{2}\right) + 0(y-0) + 1$$

$$\boxed{L(x,y) = 1.}$$

24. Find the linearization of $f(x,y) = x^2 e^y$ at the point $(1,0)$.

Answer:
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2 e^y] = 2xe^y; \quad \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,0)} = 2(1)e^0 = 2$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2 e^y] = x^2 e^y; \quad \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,0)} = (1)^2 e^0 = 1$$

$$f(1,0) = (1)^2 e^0 = 1.$$

So the linearization is

$$L(x,y) = 2(x-1) + 1(y-0) + 1$$

$$\boxed{L(x,y) = 2x + y - 1.}$$

26. Find the linear approximation of $f(x,y) = \sin(x+2y)$ at $(0,0)$ and use it to approximate $f(-0.1, 0.2)$. Using a calculator, compare the approximation with the exact value of $f(-0.1, 0.2)$.

Answer: $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}[\sin(x+2y)] = \cos(x+2y)\dfrac{\partial}{\partial x}[x+2y] = \cos(x+2y)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \cos(0 + 2(0)) = \cos 0 = 1.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin(x+2y)] = \cos(x+2y)\frac{\partial}{\partial y}[x+2y] = 2\cos(x+2y).$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = 2\cos(0+2(0)) = 2\cos 0 = 2.$$

$$f(0,0) = \sin(0+2(0)) = 0.$$

So the linearization is

$$L(x,y) = 1(x-0) + 2(y-0) + 0$$
$$= x + 2y.$$

Since $f(x,y) \approx L(x,y)$ when (x,y) is "close" to $(0,0)$,

$$f(-0.1, 0.2) \approx L(-0.1, 0.2) = (-0.1) + 2(0.2) = -0.1 + 0.4 = \boxed{0.3}.$$

My TI-89 says that

$$f(-0.1, 0.2) = \sin(-0.1 + 2(0.2)) = \sin(0.3) \approx 0.29552.$$

30. Let $\vec{f}(x,y) = \begin{bmatrix} 2x - 3y \\ 4x^2 \end{bmatrix}$. Find the Jacobi matrix of $\vec{f}$.

Answer: If $\vec{f}(x_1, \ldots, x_n) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}$ (so $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$),

the Jacobi matrix of $\vec{f}$ is

$$D\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

So here $n = 2$ and $m = 2$, and $\vec{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 4x^2 \end{bmatrix}$. So

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}[2x - 3y] = 2 \quad ; \quad \frac{\partial f_1}{\partial y} = \frac{\partial}{\partial y}[2x - 3y] = -3$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x}[4x^2] = 8x \quad ; \quad \frac{\partial f_2}{\partial y} = \frac{\partial}{\partial y}[4x^2] = 0.$$

So the Jacobi matrix is

$$D\vec{f} = \begin{bmatrix} 2 & -3 \\ 8x & 0 \end{bmatrix}.$$

32. Find the Jacobi matrix for $\vec{f}(x,y) = \begin{bmatrix} (x-y)^2 \\ \sin(x-y) \end{bmatrix}$.

Let $f_1 = (x-y)^2 = x^2 - 2xy + y^2$ and $f_2 = \sin(x-y)$. Then

$$\frac{\partial f_1}{\partial x} = 2x - 2y \quad , \quad \frac{\partial f_1}{\partial y} = 2y - 2x$$

$$\frac{\partial f_2}{\partial x} = \cos(x-y) \quad , \quad \frac{\partial f_2}{\partial y} = -\cos(x-y).$$

So the Jacobian matrix is

$$D\vec{f} = \begin{bmatrix} 2x - 2y & 2y - 2x \\ \cos(x-y) & -\cos(x-y) \end{bmatrix}.$$

40. Find a linear approximation to $\vec{f}(x,y) = \begin{bmatrix} e^x \sin y \\ e^{-y} \cos x \end{bmatrix}$ at $(0,0)$.

Answer: Let $f_1(x,y) = e^x \sin y$ and $f_2(x,y) = e^{-y} \cos x$.

Then

$$\frac{\partial f_1}{\partial x} = e^x \sin y \quad , \quad \frac{\partial f_1}{\partial y} = e^x \cos y$$

$$\frac{\partial f_2}{\partial x} = -e^{-y} \sin x \quad , \quad \frac{\partial f_2}{\partial y} = -e^{-y} \cos x$$

so the Jacobi matrix is

$$D\vec{f} = \begin{bmatrix} e^x \sin y & e^x \cos y \\ -e^{-y} \sin x & -e^{-y} \cos x \end{bmatrix}.$$

The linearization at $(0,0)$ is

$$\vec{L}(x,y) = \begin{bmatrix} f_1(0,0) \\ f_2(0,0) \end{bmatrix} + D\vec{f}(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$= \begin{bmatrix} e^0 \sin 0 \\ e^{-0} \cos 0 \end{bmatrix} + \begin{bmatrix} e^0 \sin 0 & e^0 \cos 0 \\ -e^{-0} \sin 0 & -e^{-0} \cos 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} y \\ -y \end{bmatrix} = \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

So $\boxed{\vec{L}(x,y) = \begin{bmatrix} y \\ 1-y \end{bmatrix}.}$

42. Find a linear approximation to $\vec{f}(x,y) = \begin{bmatrix} (x+y)^2 \\ xy \end{bmatrix}$ at $(-1,1)$.

Answer: $D\vec{f} = \begin{bmatrix} 2(x+y) & 2(x+y) \\ y & x \end{bmatrix}$, so

$$D\vec{f}(-1,1) = \begin{bmatrix} 2(-1+1) & 2(-1+1) \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

So the linearization is

$$\vec{L}(x,y) = \vec{f}(-1,1) + D\vec{f}(-1,1)\begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x+1 \\ y-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ x+1-y+1 \end{bmatrix}.$$

So $\boxed{\vec{L}(x,y) = \begin{bmatrix} 0 \\ x-y+1 \end{bmatrix}}$.

46. Find a linear approximation to $\vec{f}(x,y) = \begin{bmatrix} \sqrt{2x+y} \\ x-y^2 \end{bmatrix}$ at $(1,2)$. Use your result to find an approximation for $f(1.05, 2.05)$, and compare the approximation to the value of $f(1.05, 2.05)$ that you get when you use a calculator.

Answer: $D\vec{f} = \begin{bmatrix} \frac{1}{\sqrt{2x+y}} & \frac{1}{2\sqrt{2x+y}} \\ 1 & -2y \end{bmatrix}$, so

$$D\vec{f}(1,2) = \begin{bmatrix} \frac{1}{\sqrt{2(1)+2}} & \frac{1}{2\sqrt{2(1)+2}} \\ 1 & -2(2) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix}.$$

The linearization is

$$\vec{L}(x,y) = \vec{f}(1,2) + D\vec{f}(1,2)\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2(1)+2} \\ 1-2^2 \end{bmatrix} + \begin{bmatrix} 1/2 & 1/4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}x - \frac{1}{2} + \frac{1}{4}y - \frac{1}{2} \\ x - 1 - 4y + 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}x + \frac{1}{4}y - 1 \\ x - 4y + 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{4}y + 1 \\ x - 4y + 4 \end{bmatrix}.$$

So $\vec{f}(1.05, 2.05) \approx \vec{L}(1.05, 2.05) = \vec{L}(21/20, 41/20)$

$$= \begin{bmatrix} \frac{1}{2}\left(\frac{21}{20}\right) + \frac{1}{4}\left(\frac{41}{20}\right) + 1 \\ \frac{21}{20} - 4\left(\frac{41}{20}\right) + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 21/40 + 41/80 + 1 \\ 21/20 - 164/20 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 83/80 + 1 \\ -143/20 + 80/20 \end{bmatrix} = \begin{bmatrix} 163/80 \\ -63/20 \end{bmatrix}$$

$$= \begin{bmatrix} 2.0375 \\ -3.15 \end{bmatrix}.$$

From my calculator,

$$\vec{f}(1.05, 2.05) = \begin{bmatrix} \sqrt{2(1.05)+2.05} \\ 1.05 - (2.05)^2 \end{bmatrix} \approx \begin{bmatrix} 2.0372 \\ -3.1525 \end{bmatrix}.$$