110.107 - Homework #7 Solutions

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Section #1

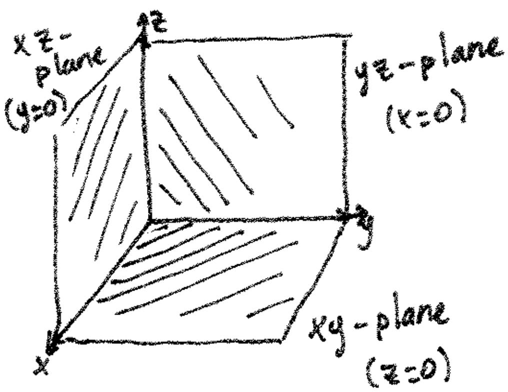
Section 10.1

4. Describe in words the set of all points in $\mathbb{R}^3$ that satisfy the following expressions:

(a) $x=0$ (b) $y=0$ (c) $z=0$ (d) $z\geq 0$ (e) $y\leq 0$

Answer: (a) The set of all points in $\mathbb{R}^3$ with $x=0$ is $(0,y,z)$, which is the yz-plane.

(b) The set of all points (x,y,z) with $y=0$ is the xz-plane.

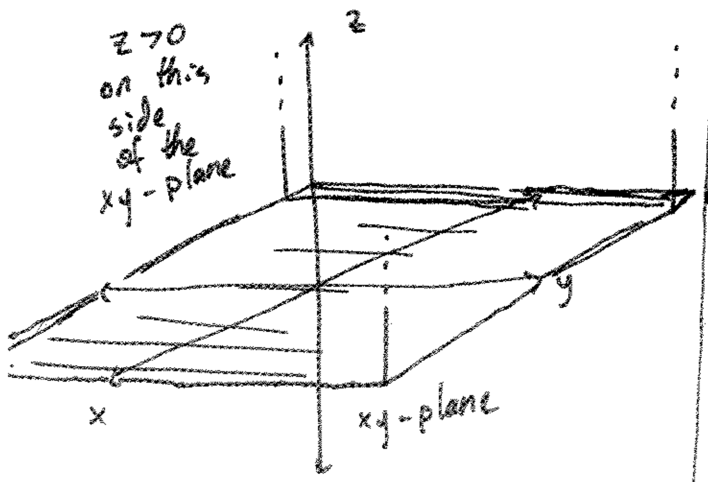


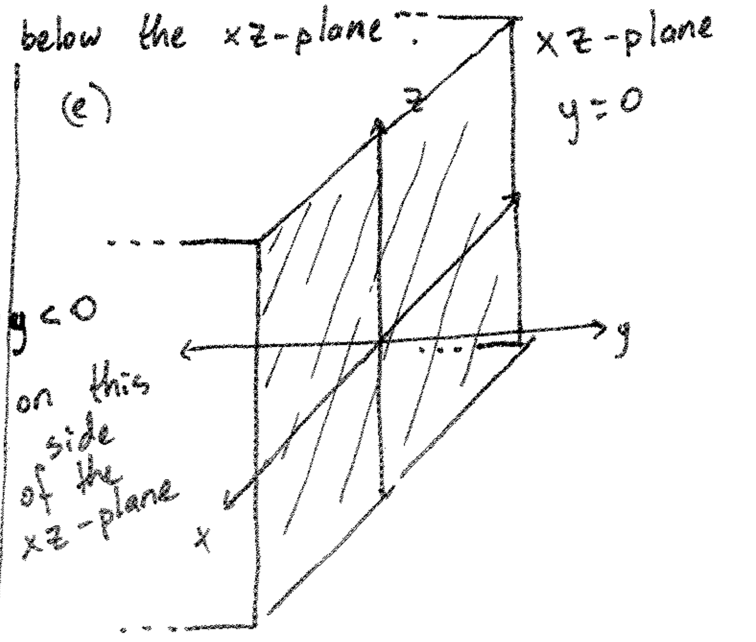
(c) The set of all points (x,y,z) with $z=0$ is the xy-plane.

(d) The set $\{(x,y,z)\in\mathbb{R}^3 : z\geq 0\}$ is all points on or above the xy-plane.

(e) The set $\{(x,y,z)\in\mathbb{R}^3 : y\leq 0\}$ is all points on or below the xz-plane.

(d)





6. Evaluate $f(x,y,z) = \sqrt{x^2 - 3y + z}$ at $(3,-1,1)$.

Answer: $f(3,-1,1) = \sqrt{3^2 - 3(-1) + 1} = \sqrt{9+3+1} = \sqrt{13}$.

So $\boxed{f(3,-1,1) = \sqrt{13}.}$

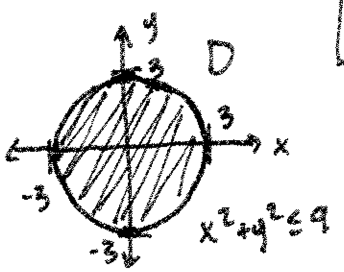
12. Evaluate $g(x_1, x_2, x_3, x_4) = x_1 x_4 \sqrt{x_2 x_3}$ at $(1, 8, 2, -1)$.

Answer: $g(1,8,2,-1) = (1)(-1)\sqrt{(8)(2)} = -1\sqrt{16} = -4$.

So $\boxed{g(1,8,2,-1) = -4.}$

14. Find the largest possible domain and the corresponding range of $f(x,y) = \sqrt{9 - x^2 - y^2}$. Determine the equation of the level curves $f(x,y) = c$, together with the possible values of c.

Answer: $f(x,y)$ is defined when $9 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 9$. $x^2+y^2=9$ is the circle of radius 3 centered at the origin, so the $\boxed{\text{domain is } D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 9\}.}$



Since $9 - x^2 - y^2 \geq 0$, the minimum value $f(x,y)$ can take occurs when $x^2+y^2=9$; so it's $f(x,y)=0$. Since $x^2+y^2 \geq 0$, the maximum value of $f(x,y)$ occurs when $x^2+y^2=0$; so $f(x,y) = \sqrt{9} = 3$. Since f is continuous it takes on all values between 0 and 3 as well. Thus the range is $\boxed{\{z \in \mathbb{R} : 0 \leq z \leq 3\}.}$ These are the only values c can take in the equation $f(x,y)=c$. The level curves are of the form

$$\sqrt{9-x^2-y^2} = c$$

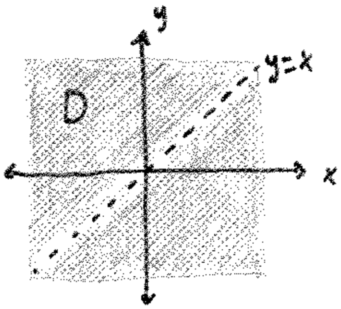
$$9 - x^2 - y^2 = c^2$$

$$x^2 + y^2 = 9 - c^2,$$

so they're $\boxed{\text{circles of radii } \sqrt{9-c^2} \text{ centered at the origin.}}$

18. Find the largest possible domain and the corresponding range of the function $f(x,y) = \frac{x+y}{x-y}$. Determine the equation of the level curves $f(x,y)=c$, along with the possible values of c.

<u>Answer</u>: The function is defined whenever $x \neq y$, so the domain is $\boxed{D = \{(x,y) \in \mathbb{R}^2 : x \neq y\}}$. To find the range, consider the values of f on the line $x-y=1$, or $y=x-1$. Then



$f(x,y) = f(x, x-1) = \frac{x+(x-1)}{x-(x-1)} = 2x-1$. As long as $y = x-1$, x can take on any value since the range of $g(x) = 2x-1$ is $(-\infty, \infty)$, the range of $f(x,y)$ is

$\boxed{\{z \in \mathbb{R} : z \in (-\infty, \infty)\}.}$ These are also all the possible values for c.

In the domain D, $x-y \neq 0$; so we can write $c = \frac{x+y}{x-y}$ as $cx - cy = x+y$, or $(c-1)x - (c+1)y = 0$.

Match each function in problems 19-22 with the graphs in figures 10.21-10.24.

19. $f(x,y) = 1+x^2+y^2$.

<u>Answer</u>: Let $y=k$ a constant. Then $f(x,k) = 1+x^2+k^2$, so the curves with constant y on the surface are parabolas that open upwards (i.e. diverge to $+\infty$). Since figures 10.21, 10.22, and 10.24 don't satisfy this criterion, $\boxed{\text{the graph of } f(x,y) = 1+x^2+y^2 \text{ is figure 10.23.}}$

20. $f(x,y) = \sin(x)\sin(y)$.

<u>Answer</u>: Let $y=k$ a constant. Then $f(x,k) = \sin(x)\sin(k)$, so the curves with constant y on the surface are sine curves. If $k=0$, then $f(x,0)=0$. Since none of 10.21, 10.23, or 10.24 have a constant value of f when $y=0$, $\boxed{\text{the graph of } f(x,y) = \sin(x)\sin(y) \text{ is figure 10.22.}}$

21. $f(x,y) = y^2 - x^2$.

Answer: Let $y = k$ be a constant; then $f(x,k) = k^2 - x^2$, so the curves of constant y are parabolas that open downwards. So the graph of f could be 10.21 or 10.24. But if we let $x = k$ a constant, then $f(k,y) = y^2 - k^2$, so the curves of constant x are parabolas that open upwards. Thus the graph of $f(x,y) = y^2 - x^2$ is figure 10.24.

22. $f(x,y) = 4 - x^2$.

Answer: There is only one figure left, so the graph of $f(x,y) = 4 - x^2$ is figure 10.21. Also, if $x = k$ a constant, then $f(k,y) = 4 - k^2$, also a constant. Figure 10.21 is the only figure in which the curves of constant x are all constant.

24. The graph in figure 10.25 shows isotherms of a lake in the temperate climate of the Northern Hemisphere.

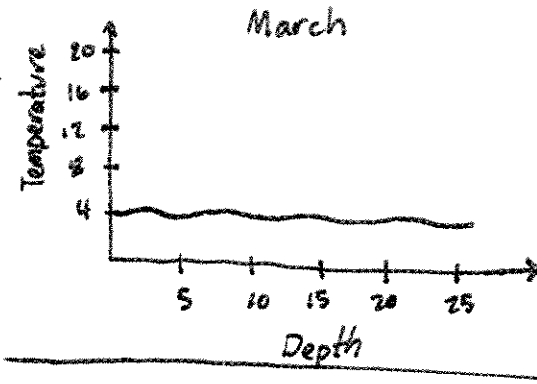
(a) Use this plot to sketch the temperature profiles in March and June. That is, plot the temperature as a function of depth for a day in March and a day in June.

(b) Explain how it follows from your temperature plots that the lake is homeothermic - that is, has the same temperature from the surface to the bottom - in March.

(c) Explain how it follows from your temperature plots that the lake is stratified - that is, has a warm layer on top (called the epilimnion), followed by a region where the temperature changes quickly (called the metalimnion), followed by a cold layer deeper down (called the hypolimnion) - in June.

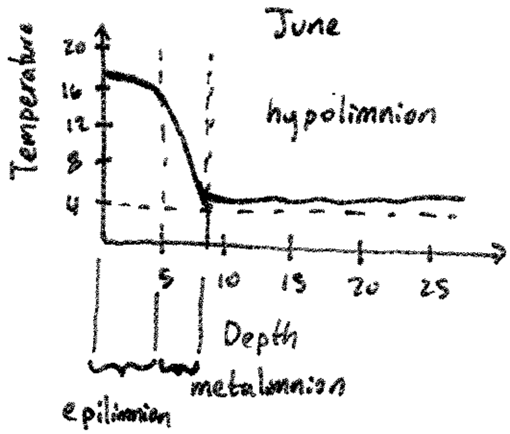
Answer: In March, the plot of temperature as a function of depth (with time held constant) lies almost exactly along the isotherm $T=4$.

So the plot looks something like this:



Clearly the lake is homeothermic, as (b) says.

In June, the temperature at the surface is somewhere between 14 - 18 degrees. Between about 5-9 meters, the temperature drops from 16 degrees to 6 degrees. Below 10 meters, the temperature is somewhere between 4 and 6 degrees. So the graph looks like this:



The epilimnion (0-5 m), metalimnion (5-9 m), and hypolimnion (10-25m) are clearly discernible on the graph, which answers (c).

Section 10.2

2. Calculate $\lim\limits_{(x,y)\to(-1,1)} (2xy+3x^2)$.

Answer: $\lim\limits_{(x,y)\to(-1,1)} (2xy+3x^2) = \left(\lim\limits_{(x,y)\to(-1,1)} 2x\right)\left(\lim\limits_{(x,y)\to(-1,1)} y\right) + \lim\limits_{(x,y)\to(-1,1)} 3x^2$

$$= (-2)(1) + 3(-1)^2 = \boxed{1.}$$

12. Calculate $\lim\limits_{(x,y)\to(-1,-2)} \dfrac{x^2-y^2}{2xy+2}$.

Answer: First, calculate the limit of the denominator:

$$\lim_{(x,y)\to(-1,-2)} (2xy+2) = \left(\lim_{(x,y)\to(-1,-2)} 2x\right)\left(\lim_{(x,y)\to(-1,-2)} y\right) + 2 = 2(-1)(-2) + 2 = 6$$

Since this limit is not zero, we can take the limit as follows:

$$\lim_{(x,y)\to(-1,-2)} \frac{x^2-y^2}{2xy+2} = \frac{\displaystyle\lim_{(x,y)\to(-1,-2)} (x^2-y^2)}{\displaystyle\lim_{(x,y)\to(-1,-2)} (2xy+2)} = \frac{\left(\displaystyle\lim_{(x,y)\to(-1,-2)} x^2\right) - \left(\displaystyle\lim_{(x,y)\to(-1,-2)} y^2\right)}{6}$$

$$= \frac{(-1)^2-(-2)^2}{6} = \frac{1-4}{6} = \boxed{-\frac{1}{2}.}$$

18. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^3}$ along lines of the form $y=mx$, for $m\neq 0$. What can you conclude?

Answer: Set $y=mx$, $m\neq 0$; then

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^3} = \lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^3} = \lim_{x\to 0} \frac{3mx^2}{x^2+m^3x^3}$$

$$= \lim_{x\to 0} \frac{3m}{1+m^3x} = 3m.$$

Since this limit depends on the value of m, the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^3}$ does not exist.

20. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y^2}{x^3+y^6}$ along lines of the form $y=mx$, $m\neq 0$, and along the parabola $x=y^2$. What can you conclude?

Answer: Let $y=mx$, $m\neq 0$. Then

$$\lim_{(x,mx)\to(0,0)} \frac{3x^2(mx)^2}{x^3+(mx)^6} = \lim_{x\to 0} \frac{3m^2x^4}{x^3+m^6x^6} = \lim_{x\to 0} \frac{3m^2x}{1+m^6x^3} = 0.$$

Now let $x = y^2$; then

$$\lim_{(y^2,y)\to(0,0)} \frac{3(y^2)^2 y^2}{(y^2)^3 + y^6} = \lim_{y\to 0} \frac{3y^6}{y^6+y^6} = \lim_{y\to 0} \frac{3y^6}{2y^6} = \frac{3}{2}.$$

Since the function approaches two different values as $(x,y) \to (0,0)$, the limit does not exist.

22. Use the definition of continuity to show that $f(x,y) = \sqrt{9+x^2+y^2}$ is continuous at $(0,0)$.

Answer: f is defined at $(0,0)$: $f(0,0) = \sqrt{9+0^2+0^2} = 3$. f is continuous at $(0,0)$ if $\lim_{(x,y)\to(0,0)} \sqrt{9+x^2+y^2} = 3$. Since $g(z) = \sqrt{z}$ is continuous when $z \geq 0$,

$$\lim_{(x,y)\to(0,0)} \sqrt{9+x^2+y^2} = \sqrt{\lim_{(x,y)\to(0,0)} (9+x^2+y^2)} = \sqrt{9 + \lim_{(x,y)\to(0,0)} (x^2) + \lim_{(x,y)\to(0,0)} (y^2)}$$

$$= \sqrt{9+0+0} = 3.$$

Thus f is continuous at $(0,0)$.

24. Show that $f(x,y) = \begin{cases} \dfrac{3xy}{x^2+y^3} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

is discontinuous at $(0,0)$. (*Hint:* Use problem 18.)

Answer: We showed in Problem 18 that $\lim_{(x,y)\to(0,0)} \dfrac{3xy}{x^2+y^3}$ does not exist. Thus it cannot equal $f(0,0) = 0$, and thus f is discontinuous at $(0,0)$.

28. (a) Write $h(x,y) = \sqrt{x+y}$ as a composition of two functions.

(b) For which values of (x,y) is $h(x,y)$ continuous?

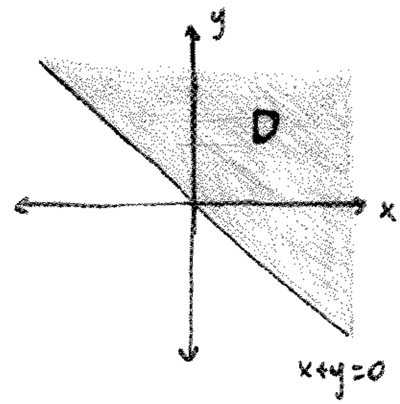
Answer: (a) Let $f(z) = \sqrt{z}$ and $g(x,y) = x+y$. Then

$(f \circ g)(x,y) = f(g(x,y)) = \sqrt{g(x,y)} = \sqrt{x+y} = h(x,y)$.

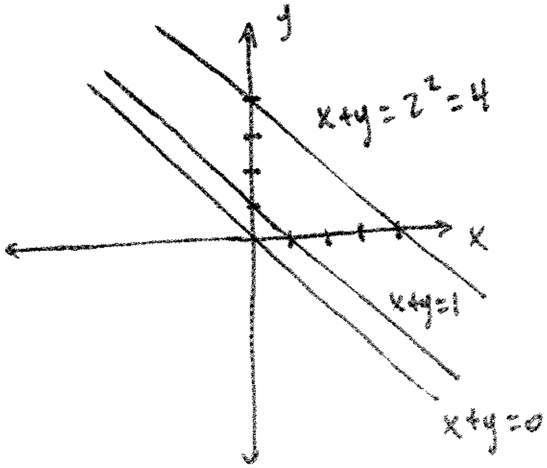
(b) The domain of $h(x,y)$ is all points (x,y) such that $x+y \geq 0$, or $y \geq -x$. Suppose $a, b \in \mathbb{R}$ such that $a+b \geq 0$; then as $f(z) = \sqrt{z}$ is continuous for $z \geq 0$, $\lim_{(x,y)\to(a,b)} \sqrt{x+y} = \sqrt{\left(\lim_{(x,y)\to(a,b)} x\right) + \left(\lim_{(x,y)\to(a,b)} y\right)} = \sqrt{a+b} = h(a,b)$, so $h(x,y)$ is continuous on its domain $D = \{(x,y) \in \mathbb{R}^2 : x+y \geq 0\}$.

EP9: Graph the domain of $h(x,y) = \sqrt{x+y}$. Graph also the c-level sets for $c = 0, 1, 2$.

Answer: The domain is $\{(x,y) \in \mathbb{R}^2 : x+y \geq 0\}$:



The level sets are $c = \sqrt{x+y}$, or $x+y = c^2$. So when $c = 0, 1, 2$, we have the graphs below:



Section 10.3

In problems 1-16, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the given functions.

6. $f(x,y) = \tan(x-2y)$.

Answer: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\tan(x-2y)] = \sec^2(x-2y)\frac{\partial}{\partial x}[x-2y] = \sec^2(x-2y)$.

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\tan(x-2y)] = \sec^2(x-2y)\frac{\partial}{\partial y}[x-2y] = -2\sec^2(x-2y)$.

10. $f(x,y) = x^2 e^{-xy/2}$.

Answer: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2 e^{-xy/2}] = e^{-xy/2}\frac{\partial}{\partial x}[x^2] + x^2\frac{\partial}{\partial x}[e^{-xy/2}]$

$= 2xe^{-xy/2} + x^2 e^{-xy/2}\frac{\partial}{\partial x}\left[-\frac{xy}{2}\right]$

$= 2xe^{-xy/2} - \frac{x^2y}{2}e^{-xy/2}$.

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2 e^{-xy/2}] = x^2 e^{-xy/2}\frac{\partial}{\partial y}\left[-\frac{xy}{2}\right] = -\frac{x^3}{2}e^{-xy/2}$.

16. $f(x,y) = \ln(3x^2 - xy)$.

Answer: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\ln(3x^2-xy)] = \frac{1}{3x^2-xy}\frac{\partial}{\partial x}[3x^2-xy] = \frac{6x-y}{3x^2-xy}$.

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\ln(3x^2-xy)] = \frac{1}{3x^2-xy}\frac{\partial}{\partial y}[3x^2-xy] = \frac{-x}{3x^2-xy} = \frac{1}{y-3x}$.

26. Let $f(x,y) = \sqrt{4-x^2-y^2}$. Compute $f_x(1,1)$ and $f_y(1,1)$, and interpret these partial derivatives geometrically.

Answer: $f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sqrt{4-x^2-y^2}\right] = \frac{1}{2\sqrt{4-x^2-y^2}}\frac{\partial}{\partial x}[4-x^2-y^2]$

$$= -\frac{x}{\sqrt{4-x^2-y^2}}.$$

So $f_x(1,1) = -\frac{1}{\sqrt{4-1-1}} = -\frac{1}{\sqrt{2}}$.

Similarly, $f_y(x,y) = \frac{\partial}{\partial y}\left[\sqrt{4-x^2-y^2}\right] = \frac{1}{2\sqrt{4-x^2-y^2}}\frac{\partial}{\partial y}[4-x^2-y^2]$

$$= -\frac{y}{\sqrt{4-x^2-y^2}}.$$

So $f_y(1,1) = -\frac{1}{\sqrt{4-1-1}} = -\frac{1}{\sqrt{2}}$.

The geometric interpretation of $f_x(x_0,y_0)$ is that it is the slope of the line tangent to the curve $f(x,y_0)$ at (x_0,y_0). There's not really much else you can say.

28. Let $f(x,y) = 2x^3 - 3yx$. Compute $f_x(1,2)$ and $f_y(1,2)$, and interpret these partial derivatives geometrically.

Answer: $f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[2x^3-3yx] = 6x^2-3y$.

So $f_x(1,2) = 6(1)^2 - 3(2) = 0$.

$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[2x^3-3yx] = -3x$.

So $f_y(1,2) = -3$.

30. Suppose that the per capita growth rate of some prey at time t depends both on the prey density $H(t)$ at t and the predator density $P(t)$ at t. Assume the relationship

$$\frac{1}{H}\frac{dH}{dt} = r\left(1 - \frac{H}{K}\right) - aP \qquad (10.2)$$

where r, K, and a are positive constants. The right-hand side of (10.2) is a function of both prey density and predator density. Investigate how an increase in (a) prey density and (b) predator density affects the per capita growth rate of this prey species.

Answer: Okay; the first thing you need to realize is that the per capita growth rate of the prey species is just $\frac{1}{H}\frac{dH}{dt}$, the change in population density with respect to time divided by the population density. So $\frac{1}{H}\frac{dH}{dt} = f(H,P)$, a function of both H and P. So what the problem is really asking for is the partial derivative of f with respect to P and to H.

(a) $\dfrac{\partial f}{\partial H} = \dfrac{\partial}{\partial H}\left[r\left(1 - \dfrac{H}{K}\right) - aP\right] = -\dfrac{r}{K}$.

Since r and K are both positive, $\frac{\partial f}{\partial H}$ is negative. This means that the per capita growth rate decreases as the population density increases. This makes sense in terms of a predator-prey system: if the prey start getting closer together, they're easier for a predator to find.

(b) $\dfrac{\partial f}{\partial P} = \dfrac{\partial}{\partial P}\left[r\left(1 - \dfrac{H}{K}\right) - aP\right] = -a$.

Thus $\frac{\partial f}{\partial P}$ is also negative, which makes sense: if the number of predators per unit area increases, it's more likely that the prey will get eaten.

34. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for $f(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$.

<u>Answer</u>: $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{xyz}{x^2+y^2+z^2}\right] = \frac{(x^2+y^2+z^2)\frac{\partial}{\partial x}[xyz] - (xyz)\frac{\partial}{\partial x}[x^2+y^2+z^2]}{(x^2+y^2+z^2)^2}$$

$$= \frac{(x^2+y^2+z^2)yz - xyz(2x)}{(x^2+y^2+z^2)^2} = \frac{(-x^2+y^2+z^2)yz}{(x^2+y^2+z^2)^2}.$$

Now, before you do <u>anything</u> else, try to guess what $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are.

I know: $\frac{\partial f}{\partial y} = \frac{(x^2-y^2+z^2)xz}{(x^2+y^2+z^2)^2}$ and $\frac{\partial f}{\partial z} = \frac{(x^2+y^2-z^2)xy}{(x^2+y^2+z^2)^2}$.

How do I know this? Symmetry: $f(x,y,z)$ is symmetric in x, y, and z.

So $f(x,y,z) = f(x,z,y) = f(y,x,z) = f(y,z,x) = f(z,x,y) = f(z,y,x)$; switch any two variables and you get the same function. That means that if you switch x and y in $f_x(x,y,z)$, you get $f_y(x,y,z)$.

This is a good trick that saves you a lot of work, but <u>it only works if your function is symmetric in the variables you're differentiating</u>.

I'll find the other two derivatives just to show you this works:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{xyz}{x^2+y^2+z^2}\right] = \frac{(x^2+y^2+z^2)\frac{\partial}{\partial y}[xyz] - xyz\frac{\partial}{\partial y}[x^2+y^2+z^2]}{(x^2+y^2+z^2)^2}$$

$$= \frac{(x^2+y^2+z^2)xz - xyz(2y)}{(x^2+y^2+z^2)^2} = \frac{(x^2-y^2+z^2)xz}{(x^2+y^2+z^2)^2}.$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{xyz}{x^2+y^2+z^2}\right] = \frac{(x^2+y^2+z^2)\frac{\partial}{\partial z}[xyz] - xyz\frac{\partial}{\partial z}[x^2+y^2+z^2]}{(x^2+y^2+z^2)^2}$$

$$= \frac{(x^2+y^2+z^2)xy - xyz(2z)}{(x^2+y^2+z^2)^2} = \frac{(x^2+y^2-z^2)xy}{(x^2+y^2+z^2)^2}.$$

And I'm right! (as usual) Pretty cool, huh?

In problems 39–48, find the indicated partial derivatives.

40. Let $f(x,y) = y^2(x-3y)$. Find $\frac{\partial^2 f}{\partial y^2}$.

<u>Answer</u>: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[y^2(x-3y)] = \frac{\partial}{\partial y}[y^2x - 3y^3] = 2yx - 9y^2$.

So $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[2yx - 9y^2\right] = \boxed{2x - 18y.}$

46. Let $f(x,y) = e^{x^2-y}$; find $\frac{\partial^3 f}{\partial y^2 \partial x}$.

<u>Answer</u>: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[e^{x^2-y}] = e^{x^2-y}\frac{\partial}{\partial x}[x^2-y] = 2xe^{x^2-y}$.

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}[2xe^{x^2-y}] = 2xe^{x^2-y}\frac{\partial}{\partial y}[x^2-y]$

$= -2xe^{x^2-y}$.

$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = \frac{\partial}{\partial y}[-2xe^{x^2-y}] = -2xe^{x^2-y}\frac{\partial}{\partial y}[x^2-y]$

$= \boxed{2xe^{x^2-y}.}$