110.107 - Homework #6 Solutions

Jake Mokris
Section #1

Section 9.4

6. Let $A = (1, 3, -2)$ and $B = (0, -1, 0)$. Find the vector representation of $\overrightarrow{AB}$.

Answer: $\overrightarrow{AB}$ is the vector starting at A and ending at B. Considering A and B as vectors, then, we know that $\vec{A} + \overrightarrow{AB} = \vec{B}$, or $\overrightarrow{AB} = \vec{B} - \vec{A}$. But $\vec{B} - \vec{A} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$, so $\boxed{\overrightarrow{AB} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}.}$

20. Use the dot product to compute the length of $[-1, 4, 3]'$.

Answer: First of all, $[-1, 4, 3]' = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$; the only reason the book uses the transpose (the prime notation) of the vector is to save space on the page. So don't be confused.

The dot product of $\vec{x} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$ with itself is

$$\vec{x} \cdot \vec{x} = \begin{bmatrix} -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = (-1)(-1) + (4)(4) + (3)(3) = 1 + 16 + 9 = 26$$

By definition, the length $|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$; thus $\boxed{|\vec{x}| = \sqrt{26}.}$

28. Let $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find $\vec{y}$ so that $\vec{x}$ and $\vec{y}$ are perpendicular.

Answer: Two vectors $\vec{x}$ and $\vec{y}$ are perpendicular if and only if $\vec{x} \cdot \vec{y} = 0$. So let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and compute $\vec{x} \cdot \vec{y}$:

$\vec{x} \cdot \vec{y} = 0$

$$\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$-2y_1 + y_2 = 0$

$y_2 = 2y_1$. Thus $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ 2y_1 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

so the solution is $\boxed{\left\{ \vec{y} \in \mathbb{R}^2 : \vec{y} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix},\ a \in \mathbb{R} \right\}}$.

30. Let $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Find $\vec{y}$ so that $\vec{x}$ and $\vec{y}$ are perpendicular.

Answer: Again, let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, and solve $\vec{x} \cdot \vec{y} = 0$:

$\vec{x} \cdot \vec{y} = 0$

$$\begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

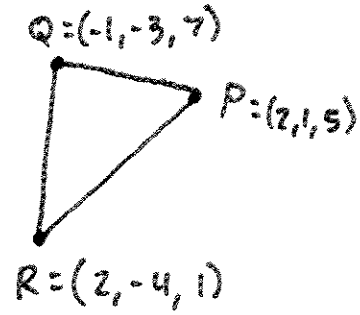
$2y_1 - y_3 = 0$

$y_3 = 2y_1$. Notice that any $y_2 \in \mathbb{R}$ satisfies this equation. So the solution is $\boxed{\left\{ \vec{y} \in \mathbb{R}^3 : \vec{y} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ a, b \in \mathbb{R} \right\}}$.

34. A triangle has vertices at coordinates $P = (2,1,5)$, $Q = (-1,-3,7)$, and $R = (2,-4,1)$.

(a) Compute the lengths of all three sides.

(b) Compute all three angles in both radians and degrees.



Answer: (a) So we have this triangle. First find the vectors that describe the sides:

$$\vec{PQ} = \vec{Q} - \vec{\mathbf{P}} = \begin{bmatrix} -1 \\ -3 \\ 7 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix}.$$

$$\vec{PR} = \vec{R} - \vec{P} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -4 \end{bmatrix}.$$

$$\vec{QR} = \vec{R} - \vec{Q} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}.$$

$$\boxed{\begin{aligned} &\text{The length of each is therefore} \\ &|\vec{PQ}| = \sqrt{(-3)^2 + (-4)^2 + 2^2} = \sqrt{29}. \\ &|\vec{PR}| = \sqrt{0^2 + (-5)^2 + (-4)^2} = \sqrt{41}. \\ &|\vec{QR}| = \sqrt{3^2 + (-1)^2 + (-6)^2} = \sqrt{46}. \end{aligned}}$$

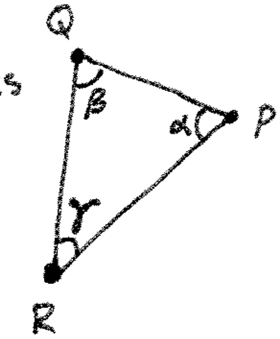
(b) Equation (9.22) on page 494 states that

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta,$$

where θ is the angle between the two vectors. Solving for θ gives

$$\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}\right)$$

Remember that the domain of $\cos^{-1}(x)$ is $x \in [-1, 1]$ and the range is $[0, \pi]$, so this formula gives you angles between 0 and π. Also note that, to get the right angles, your vectors have to be in the right direction.



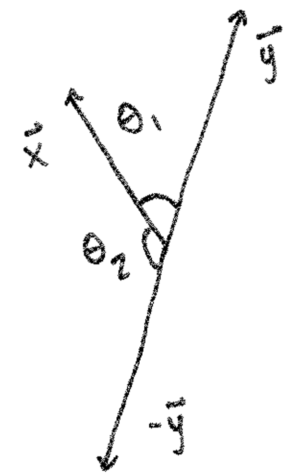
For example, if I have two vectors $\vec{x}$ and $\vec{y}$ as below, then

$$\theta_1 = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}\right),$$

but

$$\theta_2 = \cos^{-1}\left(\frac{(\vec{x}) \cdot (-\vec{y})}{|\vec{x}||-\vec{y}|}\right) = \cos^{-1}\left(-\frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}\right)$$

because $|\vec{y}| = |-\vec{y}|$.



Now we can find the angles:

$$\alpha = \cos^{-1}\left(\frac{\vec{PQ} \cdot \vec{PR}}{|\vec{PQ}||\vec{PR}|}\right) = \cos^{-1}\left(\frac{\begin{bmatrix} -3 & -4 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -5 \\ -4 \end{bmatrix}}{\sqrt{29}\sqrt{41}}\right)$$

$$= \cos^{-1}\left(\frac{20 - 8}{\sqrt{1189}}\right) = \cos^{-1}\left(\frac{12}{\sqrt{1189}}\right).$$

$$\beta = \cos^{-1}\left(\frac{\vec{QR} \cdot \vec{QP}}{|\vec{QR}||\vec{QP}|}\right) = \cos^{-1}\left(\frac{\vec{QR} \cdot (-\vec{PQ})}{|\vec{QR}||\vec{PQ}|}\right) = \cos^{-1}\left(\frac{\begin{bmatrix} 3 & -1 & -6 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}}{\sqrt{46}\sqrt{29}}\right)$$

$$= \cos^{-1}\left(\frac{9 - 4 + 12}{\sqrt{1334}}\right) = \cos^{-1}\left(\frac{17}{\sqrt{1334}}\right).$$

$$\gamma = \cos^{-1}\left(\frac{\overrightarrow{RQ}\cdot\overrightarrow{RP}}{|\overrightarrow{RQ}||\overrightarrow{RP}|}\right) = \cos^{-1}\left(\frac{\overrightarrow{QR}\cdot\overrightarrow{PR}}{|\overrightarrow{QR}||\overrightarrow{PR}|}\right)$$

$$= \cos^{-1}\left(\frac{\begin{bmatrix}3 & -1 & -6\end{bmatrix}\begin{bmatrix}0\\-5\\-4\end{bmatrix}}{\sqrt{41}\,\sqrt{46}}\right) = \cos^{-1}\left(\frac{5+24}{\sqrt{1886}}\right) = \cos^{-1}\left(\frac{29}{\sqrt{1886}}\right)$$

So the answers are

Angle	$\cos^{-1}$	Radians	Degrees
α (bet. PQ and PR)	$\cos^{-1}\left(\frac{12}{\sqrt{1189}}\right)$	≈ 1.22	$\approx 69.6^\circ$
β (bet. QR and QP)	$\cos^{-1}\left(\frac{17}{\sqrt{1334}}\right)$	≈ 1.09	$\approx 62.3^\circ$
γ (bet. RQ and RP)	$\cos^{-1}\left(\frac{29}{\sqrt{1886}}\right)$	≈ 0.840	$\approx 48.1^\circ$

Note that, if you add the three angles together, you get 3.15 when you use the approximate angle in radians, 180° when you add the approximate angles in degrees, and exactly π or 180° when you add the exact angles in a calculator. This is a way to check your answers, because you know the sum of the interior angles of a triangle must be 180°.

36. Find the equation of the line through $(3,2)$ and perpendicular to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

<u>Answer</u>: Let (x,y) be a point on the line; then the vector $\begin{bmatrix} x-3 \\ y-2 \end{bmatrix}$ is in the direction of the line. But we know that this vector is perpendicular to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so we have that

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x-3 \\ y-2 \end{bmatrix} = 0 \text{, or}$$

$$\boxed{(-1)(x-3) + (1)(y-2) = 0 \text{, or} \quad y = x-1 \text{ .}}$$

40. Find the equation of the plane through $(1,0,-3)$ and perpendicular to $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

<u>Answer</u>: Let $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{r_0} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$. Then $\vec{r} - \vec{r_0}$ is a vector in the plane, which means it's perpendicular to $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. Thus

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

$$\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y \\ z+3 \end{bmatrix} = 0$$

$$\boxed{(x-1) - 2y - (z+3) = 0 \quad \text{or} \quad x - 2y - z = 4 \text{ .}}$$

44. Find the parametric equation of the line in the xy-plane that goes through the point $(3,-4)$ in the direction of the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Answer: Suppose the point (x,y) lies on this line; then the vector from (x,y) to $(3,-4)$ is parallel to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Mathematically, this is written as $\begin{bmatrix} x-3 \\ y+4 \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ for some $t \in \mathbb{R}$.

If we write $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{r_0} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, the equation becomes $\vec{r} - \vec{r_0} = t\vec{v}$, or $\vec{r} = \vec{r_0} + t\vec{v}$

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and the parametric equations for this line are $\boxed{\{(x,y) \in \mathbb{R}^2 : x = 3 - t,\ y = -4 + 2t,\ t \in \mathbb{R}\}}$.

48. Find the parametric equation of the line in the xy-plane that goes through the points $(2,1)$ and $(3,5)$. Then eliminate the parameter to find the equation of the line in standard form.

Answer: Let $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{r_0} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Then the line is $\vec{r} = \vec{r_0} + t v$, or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, or

$\boxed{\{(x,y) \in \mathbb{R}^2 : x = 2 + t,\ y = 1 + 4t,\ t \in \mathbb{R}\}.}$ Now solve for y in terms of x: $t = x - 2$, so $y = 1 + 4(x-2)$, or $\boxed{y = 4x - 7}$.

54. Parametrize the line $x - 5y + 7 = 0$.

Answer: This is easy, and there are many ways to do it. I'll set $y = t$, and then $x = 5t - 7$. So one answer is

$$\boxed{\{(x,y) \in \mathbb{R}^2 : x = 5t - 7,\ y = t,\ t \in \mathbb{R}\} \text{ or } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix}.}$$

56. Find the parametric equation of the line in xyz-space that goes through the point $(2,0,4)$ in the direction of the vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Answer: As before, let $\vec{r}_0 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; then the parametric equation for the line is $\vec{r} = \vec{r}_0 + t\vec{u}$, or

$$\boxed{\begin{gathered} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ or} \\ \{(x,y,z) \in \mathbb{R}^3 : x = 2 + t,\ y = 2t,\ z = 4 + 3t,\ t \in \mathbb{R}\}. \end{gathered}}$$

62. Find the parametric equation of the line in xyz-space that goes through the points $(1,0,4)$ and $(3,2,0)$.

Answer: All we need is a vector parallel to the line; since $(1,0,4)$ and $(3,2,0)$ both lie on the line, $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$ works. Then the line is

$$\boxed{\begin{gathered} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}, \text{ or} \\ \{(x,y,z) \in \mathbb{R}^3 : x = 1 + 2t,\ y = 0 + 2t,\ z = 4 - 4t,\ t \in \mathbb{R}\}. \end{gathered}}$$

64. Given are (1) a plane through $(2,0,-1)$ and perpendicular to the vector $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ and (2) a line through the points $(1,0,-2)$ and $(-1,-1,1)$. Where do the plane and the line intersect?

Answer: The equation for the plane is

$$\begin{bmatrix} -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x-2 \\ y-0 \\ z+1 \end{bmatrix} = 0, \text{ or}$$

$$-x+2+y+3z+3=0, \text{ or } \quad -x+y+3z=-5.$$

The equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix},$$

or $\{(x,y,z)\in \mathbb{R}^3 : x=1+2t,\ y=t,\ z=-2-3t,\ t\in\mathbb{R}\}$.

So if (x,y,z) is the point where the line and the plane intersect, then $-x+y+3z=-5$ <u>and</u> $x=1+2t$, $y=t$, $z=-2-3t$ for some $t\in\mathbb{R}$. So we need to solve for the value of t such that $-x+y+3z=-5$:

$$-x+y+3z=-5$$

$$-(1+2t)+t+3(-2-3t)=-5$$

$$-1-2t+t-6-9t=-5$$

$$-10t=2$$

$$t=-\frac{1}{5}$$

When $t = -\frac{1}{5}$,

$x = 1 + 2\left(-\frac{1}{5}\right) = \frac{3}{5}$, $y = -\frac{1}{5}$, and $z = -2 - 3\left(-\frac{1}{5}\right) = -\frac{7}{5}$.

$$\boxed{\text{So the point where the line and the plane intersect is } \left(\frac{3}{5}, -\frac{1}{5}, -\frac{7}{5}\right).}$$