110.107 - Homework #5 Solutions

Section 9.3

2. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

(a) Show by direct calculation that $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$.

(b) Show by direct calculation that $A(\lambda\vec{x}) = \lambda(A\vec{x})$.

Answer: (a)

$$A(\vec{x}+\vec{y}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(x_1+y_1) + a_{12}(x_2+y_2) \\ a_{21}(x_1+y_1) + a_{22}(x_2+y_2) \end{bmatrix} = \begin{bmatrix} (a_{11}x_1 + a_{12}x_2) + (a_{11}y_1 + a_{12}y_2) \\ (a_{21}x_1 + a_{22}x_2) + (a_{21}y_1 + a_{22}y_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{11}y_1 + a_{12}y_2 \\ a_{21}y_1 + a_{22}y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= A\vec{x} + A\vec{y}.$$

(b)

$$A(\lambda\vec{x}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\left(\lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

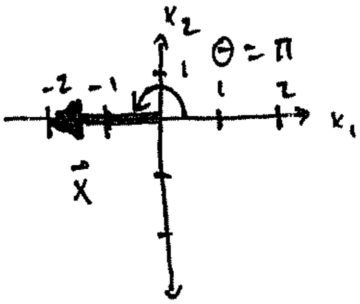
$$= \begin{bmatrix} a_{11}\lambda x_1 + a_{12}\lambda x_2 \\ a_{21}\lambda x_1 + a_{22}\lambda x_2 \end{bmatrix} = \begin{bmatrix} \lambda(a_{11}x_1 + a_{12}x_2) \\ \lambda(a_{21}x_1 + a_{22}x_2) \end{bmatrix}$$

$$= \lambda\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \lambda\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda(A\vec{x}).$$

Represent each vector in the x_1x_2-plane, and determine its length and the angle it forms with the positive x_1-axis (measured counterclockwise).

4. $\vec{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

<u>Answer</u>:

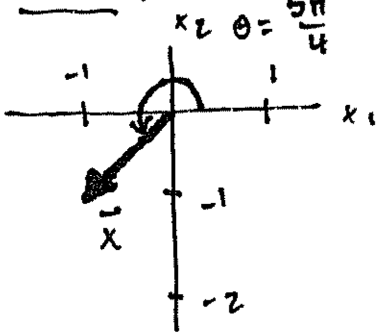


Length: $|\vec{x}| = \sqrt{(-2)^2 + 0^2} = 2$

Angle: $\theta = \pi$ (or 180°, but you should use radians)

6. $\vec{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

<u>Answer</u>:



Length: $|\vec{x}| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

Angle: $\tan\theta = \frac{-1}{-1} = 1$

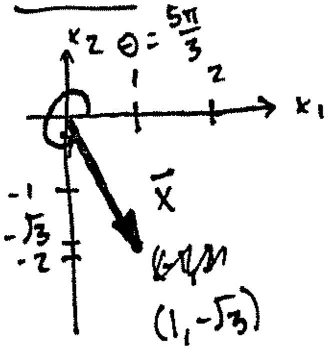
$$\theta = \tan^{-1}(1) + \pi$$
$$= \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

8. $\vec{x} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$.

Length: $|\vec{x}| = \sqrt{1 + (-\sqrt{3})^2} = 2$

Angle:

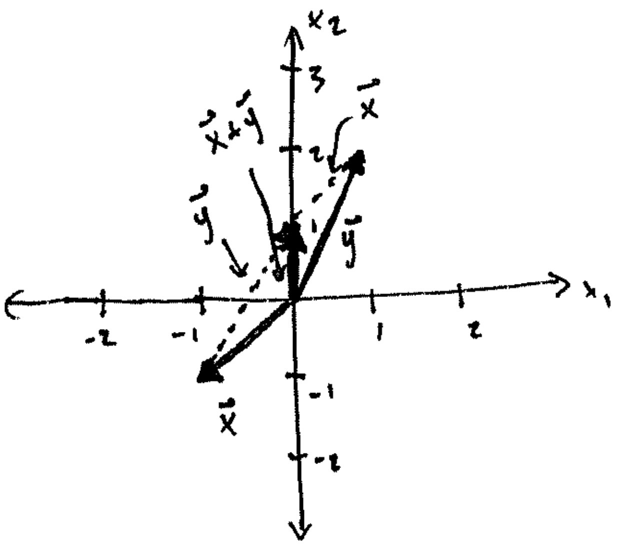
<u>Answer</u>



$$\tan\theta = -\sqrt{3}$$
$$\theta = \tan^{-1}(-\sqrt{3}) + 2\pi = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$$

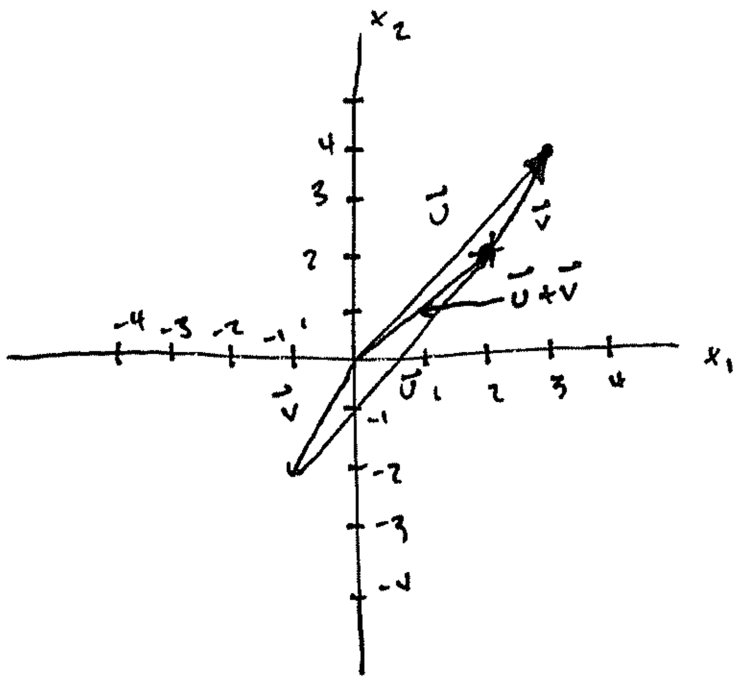
20. Find $\vec{x}+\vec{y}$, and graph $\vec{x}, \vec{y}$, and $\vec{x}+\vec{y}$.

Answer: $\vec{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\vec{x}+\vec{y} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



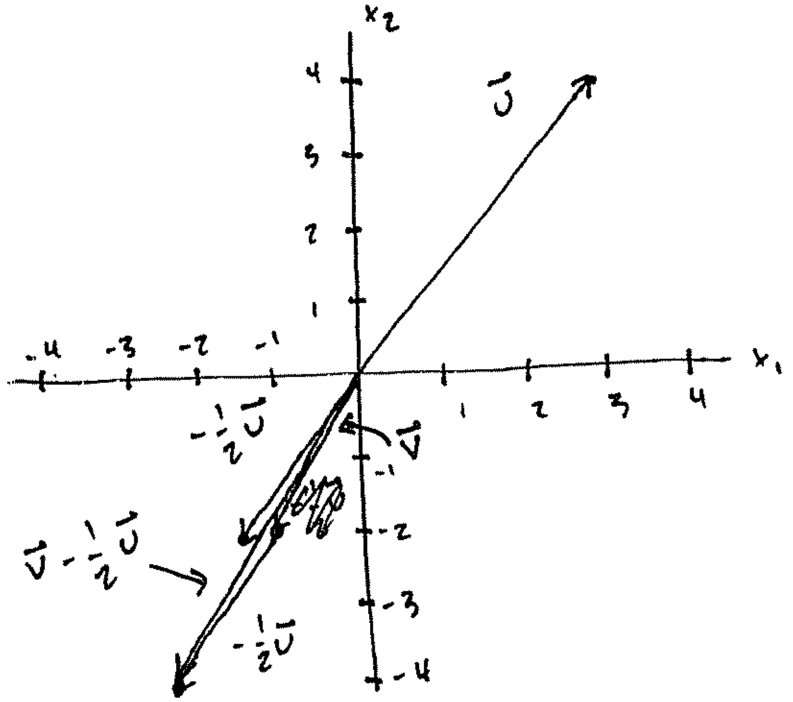
29. Let $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Compute $\vec{u}+\vec{v}$ and illustrate the result graphically.

Answer: $\vec{u}+\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 4-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.



32. Compute $\vec{v} - \frac{1}{2}\vec{u}$ and illustrate the result graphically.

Answer: $\vec{v} - \frac{1}{2}\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -4 \end{bmatrix}$



If your graphs don't look that great, well, mine don't either. The book chose vectors that are nearly parallel, which makes the parallelogram you draw quite thin.

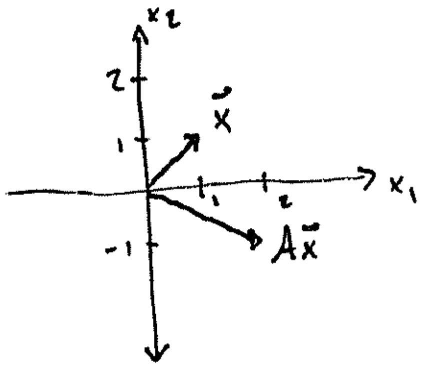
36. Give a geometric interpretation of the map $\vec{x} \to A\vec{x}$ for

$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

Answer: If $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, $A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ -b \end{bmatrix}$

So A doubles the x_1 coordinate and negates the x_2 coordinate.

For example: if $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, which looks like this:



54. Let $A = \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix}$. Find the eigenvalues and eigenvectors for A. Determine the equations for the lines through the origin in the direction of the eigenvectors, and graph the lines along with the eigenvectors and A times the eigenvectors.

Answer: First find the eigenvalues: solve the system

$$\det(A - \lambda I_2) = 0$$

$$0 = \begin{vmatrix} 3-\lambda & 6 \\ -1 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda) - (-1)(6)$$

$$= -12 - 3\lambda + 4\lambda + \lambda^2 + 6$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

So eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

Now find the eigenvectors:

Eigenvector for $\lambda_1 = -3$

$$\begin{bmatrix} 3-(-3) & 6 \\ -1 & -4-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|c} 6 & 6 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{6R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right].$$ We get the equation $x_1 + x_2 = 0$, or $x_2 = -x_1$.

So any vector of the form $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -3$. Choose $x_1 = 1$, and let $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ be our eigenvector.

Eigenvector for $\lambda_2 = 2$:

$$\left[\begin{array}{cc|c} 3-2 & 6 & 0 \\ -1 & -4-2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 6 & 0 \\ -1 & -6 & 0 \end{array}\right] \xrightarrow{R_1 + 2R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We get the equation $x_1 + 6x_2 = 0$, or $x_2 = -\frac{1}{6}x_1$. Choose $x_1 = 6$; then $v_2 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$.

Now for the graph: Since A times an eigenvector equals the eigenvector times its eigenvalue,

$$Av_1 = \lambda_1 v_1 = -3v_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

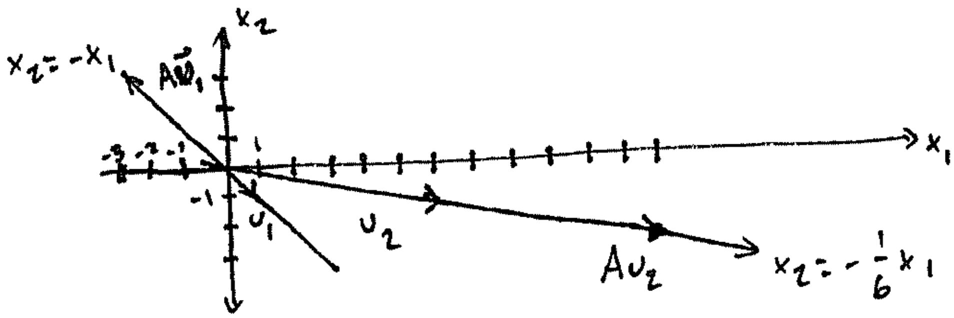
$$Av_2 = \lambda_2 v_2 = 2v_2 = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$$

The lines are simply the equations you got when you solved for the eigenvectors:

v_1: $x_2 = -x_1$

v_2: $x_2 = -\frac{1}{6}x_1$.

And here's the graph:



62. Find the eigenvalues for $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$.

<u>Answers</u>: Eigenvalues occur when $\det(A - \lambda I_2) = 0$:

$$0 = \begin{vmatrix} a-\lambda & c \\ 0 & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda).$$

So $\lambda_1 = a$ and $\lambda_2 = b$.

66. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$. Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are positive or negative.

<u>Answer</u>: Use the theorem on p. 480:

> <u>THEOREM</u> Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2. Then the real parts of λ_1 and λ_2 are negative if and only if
>
> $\operatorname{tr} A < 0$ and $\det A > 0$.

Since $\det A = \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} = (0)(-1) - (1)(2) = -2$

and $\operatorname{tr} A = 0 + (-1) = -1$, the real parts of λ_1 and λ_2 are not both negative.

70. Let $A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$.

(a) Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are eigenvectors of A and are linearly independent.

(b) Represent

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Answer:

$$A\vec{v}_1 = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2+4 \\ -4+12 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So $A\vec{v}_1 = 2\vec{v}_1$, which means $\vec{v}_1$ is an eigenvector of A with eigenvalue 2.

$$A\vec{v}_2 = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+1 \\ -4+3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So $A\vec{v}_2 = -\vec{v}_2$, which means $\vec{v}_2$ is an eigenvector of A with eigenvalue -1. The eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are linearly independent if $a_1\vec{v}_1 + a_2\vec{v}_2 = 0$ if and only if $a_1 = a_2 = 0$. So solve for a_1 and a_2:

$$a_1\begin{bmatrix} 1 \\ 4 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We know that $a_1 = a_2 = 0$ if and only if $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} \neq 0$ by the theorem on p.455. Since $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = 1 - 4 = -3 \neq 0$, $a_1 = a_2 = 0$, which means $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

(b) We want a_1 and a_2 such that

$\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$; in other words

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 4 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{-4R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -3 & 6 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2\to R_2} \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right].$$

So $a_1 = 1$ and $a_2 = -2$, which means $\vec{x} = \vec{v}_1 - 2\vec{v}_2$, or

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(c) Use your results in (a) and (b) to compute $A^{10}\vec{x}$.

<u>Answer</u>:

$$\begin{aligned} A^{10}\vec{x} &= A^{10}(\vec{v}_1 - 2\vec{v}_2) = A^{10}\vec{v}_1 - 2A^{10}\vec{v}_2 \\ &= 2^{10}\vec{v}_1 - 2(-1)^{10}\vec{v}_2 \\ &= \begin{bmatrix} 2^{10} \\ 2^{12} \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(2^9-1) \\ 2(2^{11}-1) \end{bmatrix}. \end{aligned}$$