Section 9.1

2. Solve the system $\begin{cases} 2x+3y=6 & (R_1) \\ x-4y=-4 & (R_2) \end{cases}$, graph the two lines, and use your graph to explain your solution.

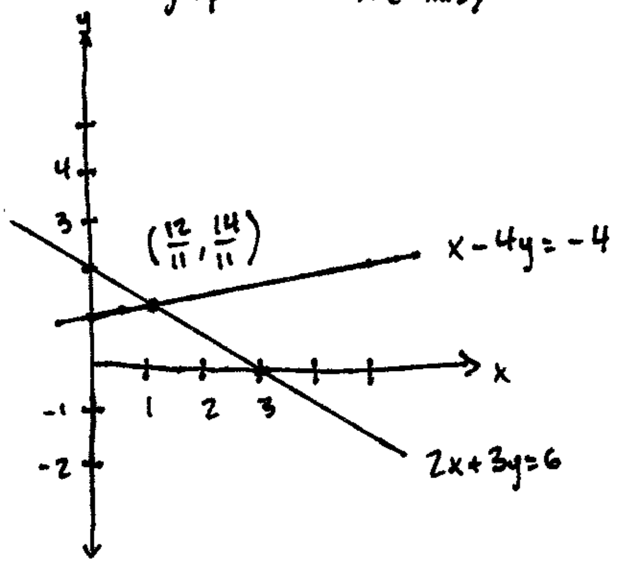
Answer: Proceed via Gaussian elimination:

$$\begin{array}{rrl} & -2x+8y=8 & -2(R_2) \\ + & 2x+3y=6 & (R_1) \\ \hline & 11y=14 & -2(R_2)+(R_1) \end{array}$$

$$\boxed{y=\frac{14}{11}}$$

Now $2x = 6-3y = 6-\frac{42}{11} = \frac{24}{11}$, so $\boxed{x=\frac{12}{11}.}$

The graph looks like this; there is clearly only one solution, with $1 < x, y < 2$, as we expected.



6. (a) Determine the solution of $\begin{cases} -2x+3y=5 & (R_1) \\ ax-y=1 & (R_2) \end{cases}$ in terms of a.

(b) For which values of a are there no solutions, exactly one solution, and infinitely many solutions?

<u>Answer</u>: (a) Proceed via Gaussian elimination:

$$\begin{array}{rrcll} & -2ax+3ay & = & 5a & a(R_1) \\ + & 2ax-2y & = & 2 & 2(R_2) \\ \hline & (3a-2)y & = & 5a+2 & \end{array}$$

$$\boxed{y=\frac{5a+2}{3a-2}}$$

Now $-2x+3y=5$, or $-2x=5-3y=5-3\left(\frac{5a+2}{3a-2}\right)=5-\frac{15a+6}{3a-2}$

$$=\frac{5(3a-2)-(15a+6)}{3a-2}=\frac{15a-10-15a-6}{3a-2}$$

$$=\frac{-16}{3a-2}$$

So $\boxed{x=\frac{8}{3a-2}.}$

(b) If $a=\frac{2}{3}$, then $3a-2=3\left(\frac{2}{3}\right)-2=2-2=0$, so when we eliminated x from the system before, we got $0y=5a+2=\frac{16}{3}$, a contradiction. So there is <u>no solution if $a=\frac{2}{3}$</u>.

If $a\neq\frac{2}{3}$, there is exactly one solution (the one we solved for above).

In order for there to be infinitely many solutions, the two lines have to be the same. But we know that, no matter what a is, $0a=0$. So when $x=0$, we get $-2(0)+3y=5$, or $y=\frac{5}{3}$ in the first line, and $a(0)-y=1$, or $y=-1$, in the second line. Since the lines have two different x-intercepts, they can never be the same. So no values of a produce infinitely many solutions.

Solve the system by reducing to upper triangular form:

10. $\begin{cases} 5x - 3y = 2 & (R_1) \\ 2x + 7y = 3 & (R_2) \end{cases}$

Answer: Use Gaussian elimination:

$$\begin{array}{rrcrl} & 10x - 6y & = & 4 & 2(R_1) \\ + & -10x - 35y & = & -15 & -5(R_2) \\ \hline & -41y & = & -11 & 2(R_1) - 5(R_2) \end{array}$$

$$\boxed{y = \frac{11}{41}.}$$

Upper triangular form: $\begin{cases} 5x - 3y = 2 & (R_1) \\ -41y = -11 & 2(R_1) - 5(R_2) \end{cases}$

So $5x = 2 + 3y = 2 + \frac{33}{41} = \frac{115}{41}$, and $\boxed{x = \frac{23}{41}.}$

16. $\begin{cases} x - 2y = 2 & (R_1) \\ 4y - 2x = -4 & (R_2) \end{cases}$

Answer: First rewrite the system: $\begin{cases} x - 2y = 2 & (R_1) \\ -2x + 4y = -4 & (R_2) \end{cases}$

Next, use Gaussian elimination:

$$\begin{array}{rrcrl} & -2x + 4y & = & -4 & (R_2) \\ + & 2x - 4y & = & 4 & 2(R_1) \\ \hline & 0 & = & 0 & 2(R_1) + (R_2) \end{array}$$

Upper triangular form: $\begin{cases} x - 2y = 2 & (R_1) \\ 0 = 0 & 2(R_1) + (R_2) \end{cases}$

The solution is

$$\boxed{\{(x, y) \in \mathbb{R}^2 : x = 2 + 2t,\ y = t,\ t \in \mathbb{R}\}.}$$

19. Show that if $a_{11}a_{22} - a_{21}a_{12} \neq 0$, then the system $\begin{cases} a_{11}x_1 + a_{12}x_2 = 0 & (R_1) \\ a_{21}x_1 + a_{22}x_2 = 0 & (R_2) \end{cases}$

has exactly one solution, namely, $x_1 = 0$ and $x_2 = 0$.

Answer: Use Gaussian elimination to solve the system:

$$\begin{array}{rll} & -a_{11}a_{21}x_1 - a_{21}a_{12}x_2 = 0 & -a_{21}(R_1) \\ + & a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = 0 & a_{11}(R_2) \\ \hline & (a_{11}a_{22} - a_{21}a_{12})x_2 = 0 & -a_{21}(R_1) + a_{11}(R_2) \\ & x_2 = 0 & \text{since } (a_{11}a_{22} - a_{21}a_{12}) \neq 0 \end{array}$$

Now use Gaussian elimination again to solve for x_1:

$$\begin{array}{rll} & -a_{11}a_{22}x_1 - a_{12}a_{22}x_2 = 0 & -a_{22}(R_1) \\ + & a_{21}a_{12}x_1 + a_{12}a_{22}x_2 = 0 & a_{12}(R_2) \\ \hline & (a_{21}a_{12} - a_{11}a_{22})x_1 = 0 & -a_{22}(R_1) + a_{12}(R_2) \\ & x_1 = 0 & \text{since } (a_{11}a_{22} - a_{21}a_{12}) \neq 0. \end{array}$$

20. Solve the system $\begin{cases} 2x - 3y + z = -1 & (R_1) \\ x + y - 2z = -3 & (R_2) \\ 3x - 2y + z = 2 & (R_3) \end{cases}$

Answer: First, use Gaussian elimination to eliminate x from R_1 and R_3:

$$\begin{array}{rll} & -2x - 2y + 4z = 6 & -2(R_2) \\ + & 2x - 3y + z = -1 & (R_1) \\ \hline & -5y + 5z = 5 & (R_1) - 2(R_2) \end{array}$$

$$\begin{array}{rll} & -3x - 3y + 6z = 9 & -3(R_2) \\ + & 3x - 2y + z = 2 & (R_3) \\ \hline & -5y + 7z = 11 & \end{array}$$

So our system becomes

$$\begin{cases} x + y - 2z = -3 & (R_4) \\ \quad -5y + 5z = 5 & (R_5) \\ \quad -5y + 7z = 11 & (R_6) \end{cases}$$

Use Gaussian elimination to eliminate y from R_6:

$$\begin{array}{rcll} 5y - 5z &=& -5 & -(R_5) \\ + \quad -5y + 7z &=& 11 & (R_6) \\ \hline 2z &=& 6 & -(R_5) + (R_6) \end{array}$$

$$\boxed{z = 3}$$

Substituting $z = 3$ into R_5 gives

$-5y = 5 - 5z = 5 - 5(3) = 5 - 15 = -10 \rightarrow \boxed{y = 2}$

Substituting $y = 2$ and $z = 3$ into R_4 gives

$x = -3 - y + 2z = -3 - 2 + 6 = 1$. So the solution is

$$\boxed{x = 1,\ y = 2,\ z = 3}.$$

26. Use the augmented matrix to solve the system $\begin{cases} 3x - 2y + z = 4 & (R_1) \\ 4x + y - 2z = -12 & (R_2) \\ 2x - 3y + z = 7 & (R_3) \end{cases}$

<u>Answer</u>: The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 4 & 1 & -2 & -12 \\ 2 & -3 & 1 & 7 \end{array}\right].$$

I will annotate my steps as follows:

$aR_i + bR_j \rightarrow R_j$ means ~~multiply~~ add a times R_i to bR_j and replace R_j with the result

$aR_i \rightarrow R_i$ means replace R_i with a times R_i

R_i / R_j means swap R_i and R_j. Here I go:

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 4 \\ 4 & 1 & -2 & -12 \\ 2 & -3 & 1 & 7 \end{array}\right] \xrightarrow[\substack{-2R_3 + R_2 \rightarrow R_2 \\ -\frac{3}{2}R_3 + R_1 \rightarrow R_1}]{} \left[\begin{array}{ccc|c} 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{13}{2} \\ 0 & 7 & -4 & -26 \\ 2 & -3 & 1 & 7 \end{array}\right] \xrightarrow[R_1 / R_3]{} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 7 \\ 0 & 7 & -4 & -26 \\ 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{13}{2} \end{array}\right]$$

$$\xrightarrow[2R_3 \to R_3]{} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 7 \\ 0 & 7 & -4 & -26 \\ 0 & 5 & -1 & -13 \end{array}\right] \xrightarrow[-\frac{5}{7}R_2 + R_3 \to R_3]{} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 7 \\ 0 & 7 & -4 & -26 \\ 0 & 0 & \frac{13}{7} & \frac{39}{7} \end{array}\right]$$

$$\xrightarrow[\frac{7}{13}R_3 \to R_3]{} \left[\begin{array}{ccc|c} 2 & -3 & 1 & 7 \\ 0 & 7 & -4 & -26 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\substack{4R_3 + R_2 \to R_2 \\ -R_3 + R_1 \to R_1}]{} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 4 \\ 0 & 7 & 0 & -14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[\frac{1}{7}R_2 \to R_2]{} \left[\begin{array}{cccc} 2 & -3 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[3R_2 + R_1 \to R_1]{} \left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_1 \to R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$ Thus $\boxed{x = -1,\ y = -2,\ z = 3.}$

30. Say whether $\begin{cases} x - y = 2 & (R_1) \\ x + y + z = 3 & (R_2) \end{cases}$ is underdetermined or overdetermined, and solve the system.

Answer: This system is underdetermined, as there are fewer equations than variables. Use Gaussian elimination to solve:

$$\begin{array}{rl} x - y = 2 & (R_1) \\ x + y + z = 3 & (R_2) \\ \hline 2x + z = 5 & (R_1) + (R_2) \\ z = 5 - 2x & \end{array}$$

From R_1, we have $y = x - 2$. So the solution is

$$\boxed{\{(x,y,z) \in \mathbb{R}^3 : x = t,\ y = t - 2,\ z = 5 - 2t,\ t \in \mathbb{R}\}.}$$

36. We have the following system: $\begin{cases} 3x_1 + 2x_2 + x_3 = 500 & (R_1) \\ 5x_1 + 3x_2 + 2x_3 = 900 & (R_2) \end{cases}$

Now solve it; I'll eliminate x_3 from R_2:

$$\begin{array}{rlr} & -6x_1 - 4x_2 - 2x_3 = -1000 & -2(R_1) \\ + & 5x_1 + 3x_2 + 2x_3 = 900 & (R_2) \\ \hline & -x_1 - x_2 = -100, & -2(R_1) + (R_2) \end{array}$$

or $x_2 = 100 - x_1$.

Substituting that into R_1 gives

$3x_1 + 2(100 - x_1) + x_3 = 500$

$x_1 + 200 + x_3 = 500$

$x_3 = 300 - x_1$. So the solution is

$$\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = t,\ x_2 = 100 - t,\ x_3 = 300 - t,\ t \in \mathbb{R} \}.$$

Now, we have to clarify our solution a little; t has to be a nonnegative integer (we can't raise half a bug, or a negative amount of bugs). So that means $t \geq 0$ and $t \leq 100$ (if $t > 100$, $x_2 < 0$).

If we add a third type of food with the bugs' respective consumption of that food, we get a third equation in our system:

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 500 & (R_1) \\ 5x_1 + 3x_2 + 2x_3 = 900 & (R_2) \\ 2x_1 + 4x_2 + x_3 = 550 & (R_3) \end{cases}$$

Now we solve again, noting that the bugs are happy because they get more food. I will use an augmented matrix, as it is easier.

Augment matrix

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 500 \\ 5 & 3 & 2 & 900 \\ 2 & 4 & 1 & 550 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 500 \\ -1 & -1 & 0 & -100 \\ -1 & 2 & 0 & 50 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3 \to R_3]{2R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 300 \\ -1 & -1 & 0 & -100 \\ -3 & 0 & 0 & 150 \end{array}\right] \xrightarrow[-\frac{1}{3}R_3 \to R_3]{-R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 300 \\ 1 & 1 & 0 & 100 \\ 1 & 0 & 0 & 50 \end{array}\right]$$

$$\xrightarrow[-R_3+R_1 \to R_1]{-R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 250 \\ 0 & 1 & 0 & 50 \\ 1 & 0 & 0 & 50 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 250 \end{array}\right]$$

So $\boxed{x_1 = 50,\ x_2 = 50,\ x_3 = 250}$.

Section 9.2

Let $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

22. Compute ABC.

Answer: $ABC = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2\cdot 1 + 3\cdot 0 & 2\cdot 2 + 3\cdot(-1) \\ (-1)\cdot 1 + 1\cdot 0 & (-1)\cdot 2 + 1\cdot(-1) \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} (-1)\cdot 2 + 0\cdot(-1) & (-1)\cdot 1 + (0)\cdot(-3) \\ 2\cdot 1 + 2\cdot(-1) & 1\cdot 1 + 2(-3) \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -2 & -1 \\ 0 & -5 \end{bmatrix}}.$$

23. Show that $AC \neq CA$.

Answer:

$$AC = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (-1)(1) + (0)(0) & (-1)(2) + (0)(-1) \\ (1)(1) + (2)(0) & (1)(2) + (2)(-1) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)(-1) + (2)(1) & (1)(0) + (2)(2) \\ (0)(-1) + (-1)(1) & (0)(0) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & -2 \end{bmatrix}.$$

So $AC \neq CA$.

30. Suppose that A is an $\ell \times p$ matrix, B is an $m \times q$ matrix, and C is an $n \times r$ matrix. What can you say about $\ell, m, n, p, q,$ and r if the products that follow are defined? State the size of the resulting matrix.

(a) ABC (b) $AB'C$ (c) BAC' (d) $A'CB'$.

Answer: If P is an $m \times n$ matrix, and Q is an $r \times s$ matrix, then PQ is defined if and only if $n = r$, and PQ is an $m \times s$ matrix. So matrix multiplication is defined only when the number of columns of the first matrix equals the number of rows in the second.

(a) Since ABC is defined, and A is $\ell \times p$, B is $m \times q$, and C is $n \times r$, $p = m$ and $q = n$. ABC is $\ell \times r$.

(b) If B is $m \times q$, then B' (the transpose of B) is $q \times m$. Since $AB'C$ is defined, we have

$$\underset{\ell \times p}{A} \quad \underset{q \times m}{B'} \quad \underset{n \times r}{C} = \underset{\ell \times r}{AB'C}, \text{ with } p = q \text{ and } m = n.$$

(c) $$\underset{m \times q}{B} \quad \underset{\ell \times p}{A} \quad \underset{r \times n}{C'} = \underset{m \times n}{BAC'}, \text{ with } q = \ell \text{ and } p = r$$

(d) $$\underset{p \times \ell}{A'} \quad \underset{n \times r}{C} \quad \underset{q \times m}{B'} = \underset{p \times m}{A'CB'}, \text{ with } \ell = n,\ r = q.$$

32. Let $A = [1 \ \ 4 \ \ -2]$, $B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Compute AB.

(b) Compute BA.

Answer: (a) A is a 1×3 matrix, and B is a 3×1 matrix, so AB is a 1×1 matrix:

$$AB = [1 \ \ 4 \ \ -2]\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = [(1)(-1) + (4)(2) + (-2)(3)] = [-1+8-6]$$

$$= [1].$$

(b)

$$BA = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}[1 \ \ 4 \ \ -2] = \begin{bmatrix} (-1)(1) & (-1)(4) & (-1)(-2) \\ (2)(1) & (2)(4) & (2)(-2) \\ (3)(1) & (3)(4) & (3)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4 & 2 \\ 2 & 8 & -4 \\ 3 & 12 & -6 \end{bmatrix}.$$

40. Write the system $\begin{cases} 2x_2 - x_1 = x_3 \\ 4x_1 + x_3 = 7x_2 \\ x_2 - x_1 = x_3 \end{cases}$.

Answer: First, put all the variables on the left side in order:

$$-x_1 + 2x_2 - x_3 = 0$$

$$4x_1 - 7x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

The matrix form of the system is therefore

$$\begin{bmatrix} -1 & 2 & -1 \\ 4 & -7 & 1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(If you're not sure, multiply the matrices together and see what you get.)

51. Suppose that $A = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$. Find X such that $AX = D$ by

(a) solving the associated system of linear equations; and

(b) using the inverse of A.

<u>Answer</u>: (a) Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; we get the equation

$\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$, which has augmented matrix

$$\left[\begin{array}{cc:c} -1 & 0 & -2 \\ 2 & -3 & -5 \end{array}\right] \xrightarrow{2R_1 + R_2 \to R_2} \left[\begin{array}{cc:c} -1 & 0 & -2 \\ 0 & -3 & -9 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_2 \to R_2]{-R_1 \to R_1} \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right].$$ So the solution is $X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(b) Use the theorem on page 454:

> <u>THEOREM</u> If
>
> $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
>
> and $a_{11}a_{22} - a_{21}a_{12} \neq 0$, then A is invertible, and
>
> $$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

So the inverse of $A = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{(-1)(-3) - (2)(0)} \begin{bmatrix} -3 & 0 \\ -2 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 & 0 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

The solution of $AX = D$ is therefore

$$X = A^{-1}D = \begin{bmatrix} -1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} (-1)(-2) + (0)(-5) \\ (-\frac{2}{3})(-2) + (-\frac{1}{3})(-5) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

58. Suppose that $A = \begin{bmatrix} a & 8 \\ 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

(a) Show that, when $a \neq 4$, $AX = B$ has exactly one solution.

(b) Suppose $a = 4$. Find conditions on b_1 and b_2 such that $AX = B$ has

(i) infinitely many solutions; and

(ii) no solutions.

(c) Explain your results in (a) and (b) graphically.

Answer: (a) If A is invertible, then $AX = B$ has exactly one solution – namely, $X = A^{-1}B$. A is invertible if and only if $\det A \neq 0$:

$$\det A = \begin{vmatrix} a & 8 \\ 2 & 4 \end{vmatrix} = (a)(4) - (8)(2) = 4a - 16.$$

So $\det A = 0$ if and only if $4a - 16 = 0$, or $a = 4$. So if $a \neq 4$, then A is invertible, and $AX = B$ has exactly one solution.

(b) If $a = 4$, we have the system

$$\begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Now consider the augmented matrix:

$$\left[\begin{array}{cc|c} 4 & 8 & b_1 \\ 2 & 4 & b_2 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 0 & -2b_2+b_1 \\ 2 & 4 & b_2 \end{array}\right]$$

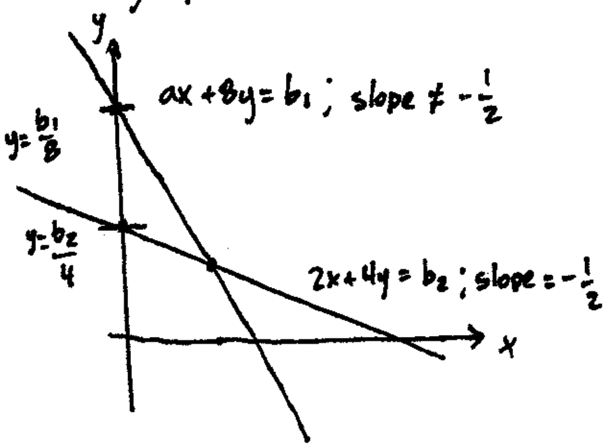
If $-2b_2+b_1=0$, then the system reduces to the equation $2x+4y=b_2$, whose solution is $\{(x,y)\in\mathbb{R}^2 : x=t,\ y=\frac{b_2-2t}{4},\ t\in\mathbb{R}\}$. Thus

(i) There are infinitely many solutions if $b_1=2b_2$.

If $-2b_2+b_1\neq 0$, then we get the equation $0x+0y=-2b_2+b_1$, or $0=-2b_2+b_1$, which is nonsense. Thus

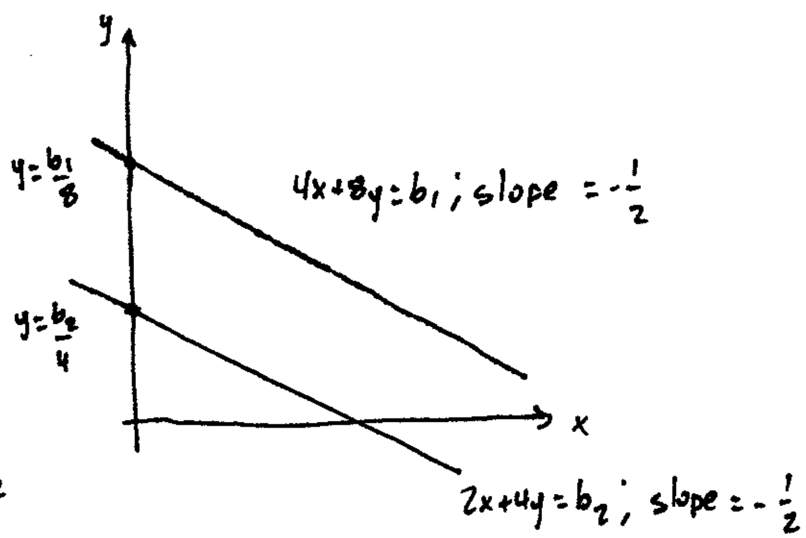
(ii) There are no solutions if $b_1\neq 2b_2$.

(c) We have the two lines $\begin{cases} ax+8y=b_1 \\ 2x+4y=b_2 \end{cases}$. If $a\neq 4$ (as in part (a)), their graph is



Since $a\neq 4$, the slope of $ax+8y=b_1$ (which is $-\frac{a}{8}$) does not equal $-\frac{1}{2}$ (the slope of $2x+4y=b_2$), so the lines intersect in one point — there is only one solution.

Now, if $a=4$, the graph looks like this:



Then the two lines are parallel. If $b_1=2b_2$, then $\frac{b_1}{8}=\frac{b_2}{4}$, which means the two lines have the same y-intercept - and are hence the same line: there are infinitely many solutions. If $b_1\neq 2b_2$, the lines never intersect, i.e., there are no solutions.

64. Use the determinant to determine whether $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ is invertible. If it is invertible, compute its inverse. In either case, ~~compute~~ solve $BX = 0$.

<u>Answer</u>: B is invertible if and only if $\det B = 0$:

$$\det B = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1)(1) - (1)(2) = -1 \neq 0.$$

So B is invertible. By the theorem on page 455, $BX = 0$ has a nontrivial solution if and only if B is singular (not invertible). Since B <u>is</u> invertible, the only solution is the trivial one: $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The inverse of B is

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

<u>Extra problems</u>

Solve the system of equations $\begin{cases} x + y = 3 \\ -y + z = 1 \\ x + z = 1 \end{cases}$.

<u>Answer</u>: The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_3 + R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{array}\right].$$ Since we have $0 = 3$, a contradiction, there is no solution.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse (if it exists) is $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Verify that $AA^{-1} = A^{-1}A = I_2$.

<u>Answer</u>: First, $\det A = ad - bc$. So

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{\det A}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

$$= \frac{1}{\det A}\begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$

And

$$A^{-1}A = \left(\frac{1}{\det A}\begin{bmatrix} +d & -b \\ -c & a \end{bmatrix}\right)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{\det A}\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$= \frac{1}{\det A}\begin{bmatrix} ad - bc & bd - bd \\ -ca + ac & -bc + ad \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$