110.107 - Homework #3 Solutions

Section 8.1

38. Assume that the size of a population, denoted by $N(t)$, evolves according to the logistic equation. Find the intrinsic rate of growth if the carrying capacity is 100, $N(0) = 10$, and $N(1) = 20$.

Answer: The logistic equation is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \quad \text{with } N(0) = N_0 \qquad (8.28)$$

where K is the carrying capacity and r is the intrinsic growth rate.
Here, $K = 100$ and $N_0 = 10$. First I will solve for N:

$$\int \frac{1}{N(K-N)} \frac{dN}{dt} dt = \int \frac{r}{K} dt \longrightarrow \int \frac{dN}{N(K-N)} = \frac{r}{K} t + C_1.$$

Now use partial fraction decomposition:

$$\frac{1}{N(K-N)} = \frac{A}{N} + \frac{B}{K-N} = \frac{A(K-N) + BN}{N(K-N)} = \frac{(B-A)N + AK}{N(K-N)}$$

This gives the system

$$\begin{cases} AK = 1 \longrightarrow A = \frac{1}{K} \\ B - A = 0 \longrightarrow A = B, \ B = \frac{1}{K} \end{cases}$$

So the integral becomes

note the negative sign

$$\int \frac{dN}{N(K-N)} = \frac{1}{K}\left(\int \frac{dN}{N} + \int \frac{dN}{K-N}\right) = \frac{1}{K}\left[\ln|N| + \left(-\ln|K-N|\right)\right]$$

$$= \frac{1}{K} \ln\left|\frac{N}{K-N}\right|.$$

Now substitute back into the differential equation:

$$\frac{1}{K} \ln\left|\frac{N}{K-N}\right| = \frac{r}{K} t + C_1 \longrightarrow \ln\left|\frac{N}{K-N}\right| = rt + C_2$$

$$\rightarrow \frac{N}{K-N} = \pm e^{rt+c_2} \longrightarrow N = C_3 e^{rt}(K-N)$$

$$\rightarrow N(1 + C_3 e^{rt}) = KC_3 e^{rt} \longrightarrow N = \frac{KC_3 e^{rt}}{1 + C_3 e^{rt}}$$

$\rightarrow N = \dfrac{KC_3}{e^{-rt} + C_3}$. We have the initial condition $N(0) = 10$, which gives

$$10 = \frac{KC_3}{1 + C_3} \longrightarrow 10(1+C_3) = KC_3 \rightarrow 10 = (K-10)C_3$$

$C_3 = \dfrac{10}{K-10} = \dfrac{10}{100-10} = \dfrac{1}{9}$, so

$$N(t) = \frac{(100)(\frac{1}{9})}{e^{-rt} + \frac{1}{9}} = \frac{100}{1 + 9e^{-rt}}.$$

Now we have that $N(1) = 20$:

$$20 = N(1) = \frac{100}{1+9e^{-r}} \longrightarrow 20(1+9e^{-r}) = 100$$

$$\rightarrow 1 + 9e^{-r} = 5 \rightarrow e^{-r} = \frac{4}{9} \longrightarrow \ln(e^{-r}) = \ln\left(\frac{4}{9}\right)$$

$$\rightarrow -r \ln e = + \ln\left(\frac{4}{9}\right) \longrightarrow \boxed{r = \ln\left(\frac{9}{4}\right)}.$$

40. Suppose that the size of a population, denoted by $N(t)$, satisfies

$$\frac{dN}{dt} = 0.7N\left(1 - \frac{N}{35}\right) \qquad (8.46)$$

(a) Determine all equilibria by solving $\frac{dN}{dt} = 0$.

Answer: First, notice that this is the logistic equation, with $r = 0.7$, $K = 35$. Equilibrium points occur when $\frac{dN}{dt} = 0$:

$$0 = \frac{dN}{dt} = 0.7N\left(1 - \frac{N}{35}\right)$$

$$N\left(1 - \frac{N}{35}\right) = 0$$

$$N(35 - N) = 0$$

Obviously equilibria occur at $N = 0$ and $N = 35$.

(b) Solve (8.46) for (i) $N(0) = 10$, (ii) $N(0) = 35$, (iii) $N(0) = 50$, and (iv) $N(0) = 0$. Find $\lim_{t\to\infty} N(t)$ for each of the four initial conditions.

Answer: First, let's solve the general logistic equation; in the solution to problem 38, we had

$$N(t) = \frac{K}{1 + e^{-rt}/C}$$

. Using the initial condition $N(0) = N_0$, we get

$$N_0 = \frac{K}{1 + 1/C}$$

$$N_0 + \frac{N_0}{C} = K$$

$$C = \frac{N_0}{K - N_0}$$

, so the solution to the logistic equation is

$$N(t) = \frac{K}{1 + e^{-rt}\left(\frac{K - N_0}{N_0}\right)} = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}.$$

In this problem, we have $r = 0.7$ and $K = 35$; so the solution becomes

$$N(t) = \frac{35}{1 + \left(\frac{35}{N_0} - 1\right) e^{-0.7t}}.$$

(i) We have that $N(0) = 10 = N_0$; so the solution is

$$N(t) = \frac{35}{1 + \left(\frac{35}{10} - 1\right) e^{-0.7t}} = \frac{35}{1 + 2.5e^{-0.7t}}.$$ Then

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{35}{1 + 2.5e^{-0.7t}} = \frac{35}{1 + 2.5 \lim_{t\to\infty} e^{-0.7t}} = 35$$

(ii) Here $N_0 = 35$. This is an equilibrium point, so $N(t) = 35$.

(iii) We have that $N(0) = 50$; the solution becomes

$$N(t) = \frac{35}{1 + \left(\frac{35}{50} - 1\right) e^{-0.7t}} = \frac{35}{1 - 0.3e^{-0.7t}}.$$

Since $t \geq 0$, $e^{-0.7t} \leq 1$, and $0.3e^{-0.7t} \leq 0.3$.

Thus $1 - 0.3e^{-0.7t} \geq 1 - 0.3 = 0.7 > 0$ for all $t \geq 0$, which means the denominator never equals zero. So we can take a limit as usual:

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{35}{1 - 0.3e^{-0.7t}} = \frac{35}{1 - 0.3 \lim_{t\to\infty} e^{-0.7t}} = 35.$$

(iv) When $N(0) = 0$, the system is in equilibrium; thus $N(t) = 0$.

<u>Solve the following three differential equations:</u>

46. $\frac{dy}{dx} = \frac{y}{x+1}$, $y(0) = 1$.

<u>Answer</u>: Use separation of variables:

$$\int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{dx}{x+1}$$

$$\int \frac{dy}{y} = \ln|x+1| + C_1$$

$\ln|y| = \ln|x+1| + c_1$

$e^{\ln|y|} = e^{c_1} e^{\ln|x+1|}$

$|y| = e^{c_1}|x+1|$

$y = \pm e^{c_1}|x+1| = c|x+1|$

Since the initial time is $x = 0$, we assume that $x \geq 0$, so that $|x+1| = x+1$. So $y = c(x+1)$; the initial condition gives

$1 = y(0) = c(0+1)$. So $c = 1$, and the solution is $\boxed{y = x+1.}$

50. $\frac{du}{dt} = \frac{\sin t}{u^2+1}$, $u(0) = 3$.

Answer: Again, use separation of variables:

$\int (u^2+1)\frac{du}{dt}\,dt = \int \sin t\,dt$

$\int (u^2+1)\,du = -\cos t + C$

$\frac{1}{3}u^3 + u = -\cos t + C$

Here we cannot easily solve for u; we can leave the equation in implicit form and solve for C:

$\frac{1}{3}(3)^3 + 3 = -\cos 0 + C$

$9 + 3 = -1 + C$

$C = 13$. So the solution is

$\boxed{\frac{1}{3}u^3 + u = -\cos t + 13.}$

52. $\frac{dx}{dy} = \frac{1}{2}\frac{x}{y}$, $x(3) = 2$.

Answer: The book is deliberately trying to confuse you by making x a function of y; note that we're solving for x:

$$\int \frac{1}{x}\frac{dx}{dy}\,dy = \frac{1}{2}\int \frac{dy}{y}$$

$$\int \frac{dx}{x} = \frac{1}{2}\ln|y| + c_1$$

$$\ln|x| = \frac{1}{2}\ln|y| + c_1$$

Again, assume $y \geq 3$, so that $\ln|y| = \ln y$; the equation becomes

$$\ln|x| = \frac{1}{2}\ln y + c_1$$

$$e^{\ln|x|} = e^{\ln y^{1/2}} e^{c_1}$$

$$|x| = e^{c_1}\sqrt{y}$$

$$x = \pm e^{c_1}\sqrt{y}$$

$x = c\sqrt{y}$. Now use the initial condition:

$2 = c\sqrt{3} \rightarrow c = \frac{2}{\sqrt{3}}$; thus

$$\boxed{x = \frac{2}{\sqrt{3}}\sqrt{y}}.$$

Section 8.2

2. Suppose that

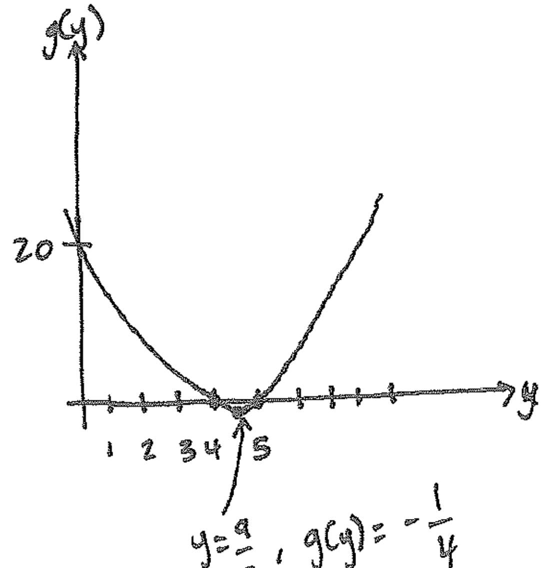
$$\frac{dy}{dx} = (4-y)(5-y).$$

(a) Find the equilibria of this differential equation.

<u>Answer</u>: Equilibria occur when $\frac{dy}{dx} = 0$, so $0 = (4-y)(5-y)$. Clearly the equilibria are $y = 4$ and $y = 5$.

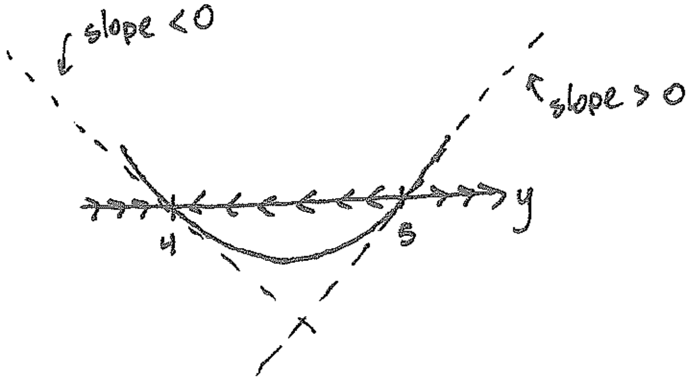
(b) Graph $\frac{dy}{dx}$ as a function of y, and use your graph to discuss the stability of the equilibria.

<u>Answer</u>: Write $\frac{dy}{dx} = g(y) = (4-y)(5-y) = 20 - 9y + y^2$. Then $\frac{dg}{dy} = 2y - 9$, and the graph has one critical point, at $0 = 2y - 9$ $\rightarrow y = \frac{9}{2}$. Since $\frac{d^2g}{dy^2} = 2 > 0$, $y = \frac{9}{2}$ is a minimum. So the graph has a minimum value of $g\left(\frac{9}{2}\right) = 20 - \frac{81}{2} + \frac{81}{4}$ $= \frac{80}{4} - \frac{81}{4} = -\frac{1}{4}$; the graph looks like this:



(pretend it's more symmetric and to scale)

Now I'll draw another graph, this time around the interval $y \in [4,5]$:



Consider the equilibrium at $y=4$. If $y<4$, then $g(y)=\frac{dy}{dx}>0$; thus, as x increases, y increases and moves toward $y=4$. If $4<y<5$, then $g(y)=\frac{dy}{dx}<0$, so as x increases, y decreases and moves toward $y=4$. Thus $y=4$ is a stable equilibrium. Think about a particle sitting at $y=4$ that obeys the differential equation (y its position and x time). If the particle were displaced slightly from $y=4$, it will move back as time progresses. (the arrows on the axis represent which way the particle will move)

Now consider what happens at $y=5$; we already saw that if $4<y<5$, $\frac{dy}{dx}=g(y)<0$, so y decreases to 4 as x increases; thus y moves away from 5. If $y>5$, $g(y)=\frac{dy}{dx}>0$, so y increases as x increases; i.e. y moves away from 5. Thus $y=5$ is an unstable equilibrium. Again, think about the particle: if it sits at $y=5$ and then is moved slightly, it will not return to $y=5$.

Note that, at $y=4$, the stable equilibrium, the slope of the tangent to $g(y)$ (the derivative of g at $y=4$) is negative. And at $y=5$, the unstable equilibrium, the derivative of g is positive. This is true in general, and is explained in the book.

(c) Compute the eigenvalues associated with each equilibrium, and discuss the stability of the equilibria.

<u>Answer</u>: The book derives the test we'll use to classify the equilibria of differential equations. The derivation is on page 409.

But we will just use the result:

Stability Criterion Consider the differential equation

$$\frac{dy}{dx} = g(y)$$

where $g(y)$ is a differentiable function. Assume that $\hat{y}$ is an equilibrium; that is, $g(\hat{y}) = 0$. Then

$$\hat{y} \text{ is locally stable if } g'(\hat{y}) < 0$$

$$\hat{y} \text{ is unstable if } g'(\hat{y}) > 0$$

(When $g'(\hat{y}) = 0$, this criterion cannot be used.)

The "eigenvalue" is just $g'(\hat{y})$. Since $g(y) = 20 - 9y + y^2$, $g'(y) = 2y - 9$. Then $g'(4) = 8 - 9 = -1$; thus $y = 4$ is a stable equilibrium by the criterion above. Also, $g'(5) = 10 - 9 = 1 > 0$, so $y = 5$ is an unstable equilibrium. Note that these are the same conclusions we made in part (b).

4. Suppose that

$$\frac{dy}{dx} = y(2-y)(y-3).$$

(a) Find the equilibria of this differential equation.

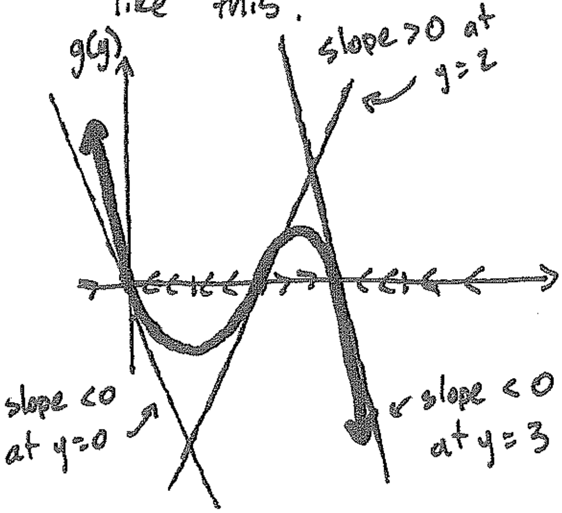
Answer: Equilibria occur when $\frac{dy}{dx} = 0$; thus the equilibria are $y = 0$, $y = 2$, and $y = 3$.

(b) Graph $\frac{dy}{dx}$ as a function of y, and use your graph to discuss the stability of the equilibria.

Answer: Let $g(y) = y(2-y)(y-3)$; multiplying out gives

$$g(y) = y(2-y)(y-3) = y(2y - 6 - y^2 + 3y) = y(-y^2 + 5y - 6)$$
$$= -y^3 + 5y^2 - 6y$$

I'm not going to bother too much finding critical points; since $\lim_{y\to\infty} g(y) = -\infty$ and $\lim_{y\to -\infty} g(y) = \infty$, the graph has to look something like this:



At $y=0$, we see that $g(y) = \frac{dy}{dx} > 0$ to the left and $g(y) = \frac{dy}{dx} < 0$ when $y > 0$. Thus, a particle directly to the left of $y=0$ will move towards $y=0$, as will a particle slightly to the right of $y=0$. Thus $y=0$ is a stable equilibrium. At $y=2$, $\frac{dy}{dx} < 0$ when y is close to 2 but less than 2. $\frac{dy}{dx} > 0$ when y is near but greater than 2. So as x increases, if $y < 2$, y will decrease; if $y > 2$, y will increase. Thus $y=2$ is an unstable equilibrium. At $y=3$, $\frac{dy}{dx} > 0$ when $y \in (2,3)$, and $\frac{dy}{dx} < 0$ when $y > 3$. Thus, as x increases, y increases if $y < 3$ and decreases if $y > 3$. So $y=3$ is a stable equilibrium.

(c) Compute the eigenvalues associated with each equilibrium, and discuss the stability of the equilibria.

<u>Answer</u>: $g'(y) = -3y^2 + 10y - 6$; the stability criterion gives

equilibrium point	eigenvalue	stability
$y=0$	$g'(0) = -3(0)^2 + 10(0) - 6$ $= -6 < 0$	stable
$y=2$	$g'(2) = -3(2)^2 + 10(2) - 6$ $= -12 + 20 - 6$ $= 2 > 0$	unstable
$y=3$	$g'(3) = -3(3)^2 + 10(3) - 6$ $= -27 + 30 - 6$ $= -3 < 0$	stable

20. Levins Model Denote by $p = p(t)$ the faction of occupied patches in a metapopulation model, and assume that

$$\frac{dp}{dt} = 2p(1-p) - p \quad \text{for } t \geq 0 \qquad (8.68)$$

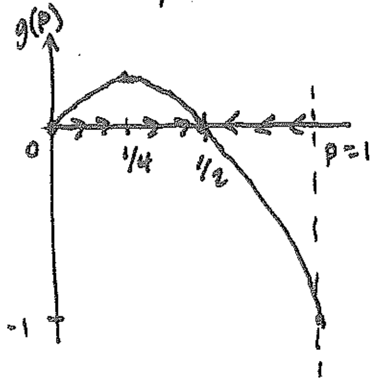
(a) set $g(p) = 2p(1-p) - p$. Graph $g(p)$ for $p \in [0,1]$.

Answer: First, note that $g(p) = 2p(1-p) - p = p[2(1-p)-1] = p(1-2p)$.

So $g(p)$ is a parabola with $\lim_{p \to \infty} g(p) = \lim_{p \to -\infty} g(p) = -\infty$. Also,

$g'(p) = p(-2) + (1-2p) = 1-4p$, so the maximum of $g(p)$ occurs at $p = \frac{1}{4}$.

The equilibria are $p=0$ and $p=\frac{1}{2}$, and the graph looks like this:



(b) Find all equilibria in (8.68) that are in $[0,1]$. Use your graph in (a) to determine their stability.

Answer: We already saw that the equilibria (where $g(p)=0$) are at $p=0$ and $p=\frac{1}{2}$. First consider the equilibrium at $p=0$: When $p<0$, $g(p) = \frac{dp}{dt} < 0$, so as t increases, p decreases, moving away from $p=0$. When $p \in (0, \frac{1}{2})$, $g(p) = \frac{dp}{dt} > 0$, so as t increases, p increases, moving away from $p=0$. Thus $p=0$ is an unstable equilibrium. Now consider the equilibrium at $p=\frac{1}{2}$: when $p \in (0,\frac{1}{2})$, $g(p) = \frac{dp}{dt} > 0$, so as t increases, p increases toward $p = \frac{1}{2}$. When $p > \frac{1}{2}$, $g(p) = \frac{dp}{dt} < 0$, so as t increases, p decreases to $p=\frac{1}{2}$. Thus $p=\frac{1}{2}$ is a stable equilibrium.

(c) Use the eigenvalue approach to analyze the stability of the equilibria that you found in (b).

Answer: We already have $g'(p)$; I'll put the table on the next page.

Equilibrium point	eigenvalue	stability
$p = 0$	$g'(0) = 1 - 4(0) = 1 > 0$	unstable
$p = \frac{1}{2}$	$g'(\frac{1}{2}) = 1 - 4(\frac{1}{2}) = -1 < 0$	stable

24. Allee effect Denote the size of a population at time t by $N(t)$, and assume that

$$\frac{dN}{dt} = 2N(N-10)\left(1 - \frac{N}{100}\right) \quad \text{for } t \geq 0. \qquad (8.72)$$

(a) Find all equilibria of (8.72)

Answer: Equilibria occur when $\frac{dN}{dt} = 0$, and so are the zeros of $g(N) = 2N(N-10)\left(1 - \frac{N}{100}\right)$. So the equilibria are $N = 0$, $N = 10$ and $N = 100$.

~~Set~~

$$\cancel{g(N) = 2N(N-10)\left(1 - \frac{N}{100}\right)}$$

~~Set $N = 0$ and graph $g(N)$~~

(b) Use the eigenvalue approach to determine the stability of the equilibria you found in (a).

Answer: First multiply out $g(N)$:

$$g(N) = 2N(N-10)\left(1 - \frac{N}{100}\right) = (2N^2 - 20N)\left(1 - \frac{N}{100}\right) = 2N^2 - \frac{2N^3}{100} - 20N + \frac{20N^2}{100}$$

$$= -\frac{N^3}{50} + \frac{11}{5}N^2 - 20N.$$

Thus $g'(N) = -\frac{3}{50}N^2 + \frac{22}{5}N - 20$. Now we can make the table (again, on the next page).

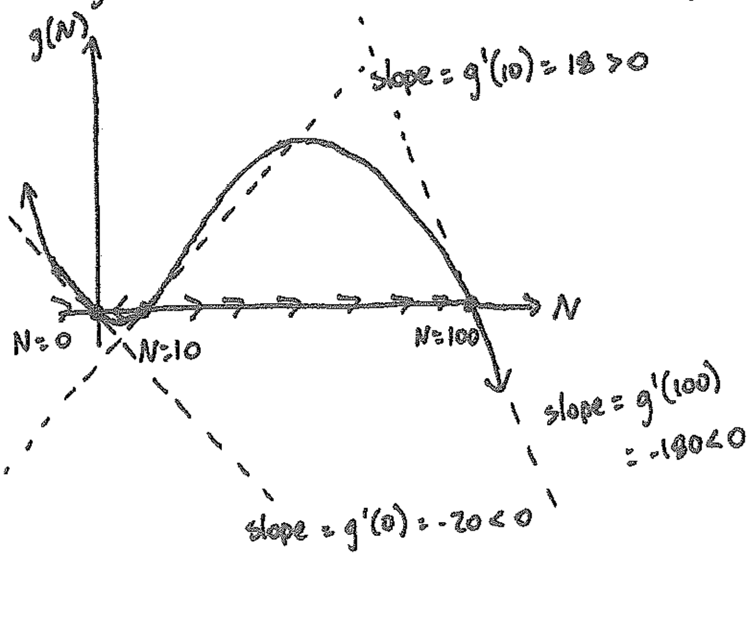
Equilibrium point	Eigenvalue	Stability
$N=0$	$g'(0)=-\frac{3}{50}(0)^2+\frac{22}{5}(0)-20=-20<0$	stable
$N=10$	$g'(10)=-\frac{3}{50}(10)^2+\frac{22}{5}(10)-20=-6+44-20$ $=18>0$	unstable
$N=100$	$g'(100)=-\frac{3}{50}(100)^2+\frac{22}{5}(100)-20$ $=-\frac{30000}{50}+\frac{2200}{5}-20$ $=-600+440-20=-180<0$	stable

(c) Set

$$g(N)=2N(N-10)\left(1-\frac{N}{100}\right)$$

for $N\geq 0$, and graph $g(N)$. Identify the equilibria of (8.72) on your graph, and use the graph to determine the stability of the equilibria. Compare your results with your findings in (b). Use your graph to give a graphical interpretation of the eigenvalues associated with the equilibria.

<u>Answer</u>: First, note that $\lim_{N\to\infty} g(N)=-\infty$ and $\lim_{N\to-\infty} g(N)=\infty$. Again, I will ignore the critical points of $g(N)$, as they are not relevant to the problem. The graph looks like this (or something like it):



First consider the equilibrium at $N=0$: when $N<0$, $g(N)=\frac{dN}{dt}>0$, so as t increases, N increases towards 0. When $N\in(0,10)$, $g(N)=\frac{dN}{dt}<0$, so as t increases, N decreases towards 0. Thus 0 is a stable equilibrium.

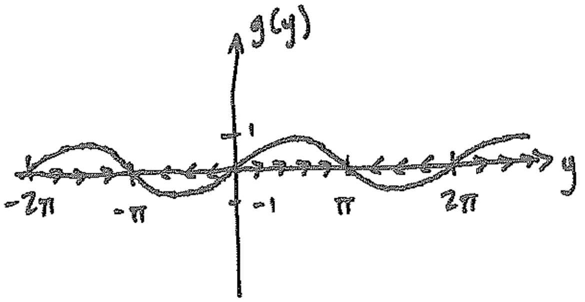
The equilibrium at $N=10$ is unstable: $\frac{dN}{dt}<0$ when $N\in(0,10)$ and $\frac{dN}{dt}>0$ when $N\in(10,100)$.

The equilibrium at $N=100$ is stable: $\frac{dN}{dt}>0$ when $N\in(10,100)$, and $\frac{dN}{dt}<0$ when $N>100$.

All these conclusions agree with our analysis in (b). Finally, a "graphical interpretation of the eigenvalues associated with the equilibria" just refers to the fact that $g'(N_0)$ is the slope of the tangent line to g at N_0.

EP4: Find a stable equilibrium solution to the autonomous differential equation $\frac{dy}{dx} = \sin y$ and show that it is stable.

Answer: Let $g(y) = \sin y$; we want a point where $\sin y = 0$. Look at the graph of $g(y)$:



equilibria occur whenever $y = n\pi$, $n \in \mathbb{Z}$ ($\mathbb{Z}$ means integers). But we want a stable equilibrium, where $g'(n\pi) < 0$.

$g'(y) = \cos y$, so $g'(n\pi) = \cos(n\pi) = \pm 1$.

So all points $y = n\pi$ where $\cos(n\pi) = -1$ are stable equilibria. This happens whenever n is odd. So the stable equilibria of our differential equation are $\pi, 3\pi, 5\pi, 7\pi, \ldots$ and $-\pi, -3\pi, -5\pi, -7\pi, \ldots$ and so on.

EP5: Determine all equilibrium solutions to the differential equation $\frac{dy}{dx} = y(1-y)^2(2-y)$. Describe the stability of each equilibrium.

Answer: Equilibria occur when $\frac{dy}{dx} = g(y) = 0$, which happens, in this case, when $y = 0$, $y = 1$, or $y = 2$. Now multiply out $g(y)$ so we can find $g'(y)$:

$$g(y) = y(1-y)^2(2-y) = y(y^2-2y+1)(2-y) = y(2y^2-y^3-4y+2y^2+2-y)$$

$$= y(-y^3+4y^2-5y+2) = -y^4+4y^3-5y^2+2y$$

$$g'(y) = -4y^3+12y^2-10y+2$$

Now, as usual, I will make a table:

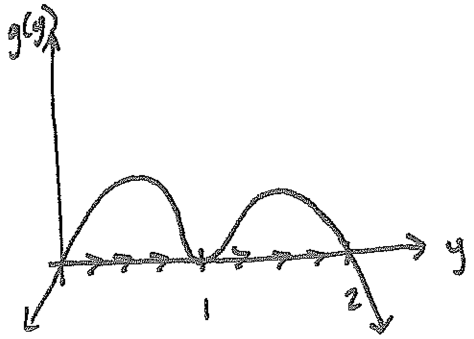
Equilibrium point	Eigenvalue	Stability
$y=0$	$g'(0) = -4(0)^3 + 12(0)^2 - 10(0) + 2 = 2 > 0$	unstable
$y=1$	$g'(1) = -4(1)^3 + 12(1)^2 - 10(1) + 2$ $= -4 + 12 - 10 + 2 = 0$	? ? ?
$y=2$	$g'(2) = -4(2)^3 + 12(2)^2 - 10(2) + 2$ $= -32 + 48 - 20 + 2 = -2 < 0$	stable

So what happens at $y=1$? Well, let's look at $g''(1)$:

$$g''(y) = -12y^2 + 24y - 10$$

$$g''(1) = -12(1)^2 + 24(1) - 10 = 2 > 0.$$

Since $g''(1) > 0$, $g'(y)$ is increasing at $y=1$, and hence $g(y)$ has a local minimum at $y=1$. So the graph looks like this (or something like it):



$g(y) = \frac{dy}{dx} > 0$ when $y<1$ and when $y<1$.

So if $y<1$, as t increases, y increases towards 1. If $y>1$, as t increases, y increases <u>away</u> from 1. This is called a <u>semistable</u> equilibrium.

<u>EP6</u>: For the logistic equation $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$, do the following:

(a) Show that, by using the change of variables $M = \frac{N}{K}$, you can rewrite the differential equation as $\frac{dM}{dt} = rM(1-M)$.

<u>Answer</u>: Let $M = \frac{N}{K}$; then $\frac{dM}{dt} = \frac{dM}{dN}\frac{dN}{dt} = \frac{1}{K}\frac{dN}{dt}$ by the chain rule, and the differential equation becomes

$$K\frac{dM}{dt} = r(KM)(1-M) \rightarrow \frac{dM}{dt} = rM(1-M).$$

(b) Use the partial fraction method to solve $\frac{dM}{dt} = rM(1-M)$.

Answer: We get that

$$\int \frac{1}{M(1-M)} \frac{dM}{dt}\, dt = \int r\, dt$$

$$\int \frac{dM}{M(1-M)} = rt + C_1$$

Now use partial fraction decomposition:

$$\frac{1}{M(1-M)} = \frac{A}{M} + \frac{B}{1-M} = \frac{A(1-M) + BM}{M(1-M)} = \frac{(B-A)M + A}{M(1-M)}$$

So $\begin{cases} B - A = 0, \Rightarrow B = A \\ A = 1, \Rightarrow B = 1 \end{cases}$ and the integral becomes

$$\int \frac{dM}{M(1-M)} = \int \frac{dM}{M} + \int \frac{dM}{1-M} = \ln|M| - \ln|1-M| = \ln\left|\frac{M}{1-M}\right|.$$

Substituting back in gives

$$\ln\left|\frac{M}{1-M}\right| = rt + C_1$$

$$\left|\frac{M}{1-M}\right| = e^{C_1} e^{rt}$$

$$\frac{M}{1-M} = \pm e^{C_1} e^{rt} = Ce^{rt}$$

$$M = (1-M)Ce^{rt}$$

$$M(1 + Ce^{rt}) = Ce^{rt}$$

$$\boxed{M = \frac{Ce^{rt}}{1 + Ce^{rt}} = \frac{1}{1 + e^{-rt}/C}}$$

(c) Suppose the model in part (b) were used to model a lake recently restocked with fish, where M is measured in thousands of fish. What is the long-term population size for the lake if the initial restocking used 200 fish?

Answer: We know the initial population size: 200. So we need the initial value for M. The problem says that M is measured in thousands of fish; so if $M=1$, there are 1000 fish, i.e. $N=1000$. Since $M=\frac{N}{K}$, we have that $1=\frac{1000}{K}$, or $K=1000$. Thus $M(0)=\frac{N(0)}{K}=\frac{200}{1000}=\frac{1}{5}$.

Now we can solve for M and C:

$$\frac{1}{5}=M(0)=\frac{1}{1+e^{-r(0)}/c} \longrightarrow \frac{1}{5}=\frac{1}{1+1/c}$$

$$1+\frac{1}{c}=5 \longrightarrow \frac{1}{c}=4, \text{ or } C=\frac{1}{4}.$$

So $M(t)=\dfrac{1}{1+4e^{-rt}}$. The long-term population size is the value that KM approaches as $t\to\infty$:

$$\lim_{t\to\infty} KM(t)=K\lim_{t\to\infty}\frac{1}{1+4e^{-rt}}=\frac{K}{1+4\lim_{t\to\infty}e^{-rt}}=\frac{K}{1+0}=K \text{ (if } r>0)$$

$$=1000.$$

So the population approaches 1000 fish as time goes on. (Note: I assumed $r>0$. If $r<0$, then $\lim_{t\to\infty} e^{-rt}=\infty$, and so $\lim_{t\to\infty} KM(t)=0$; in other words, the fish die out.)