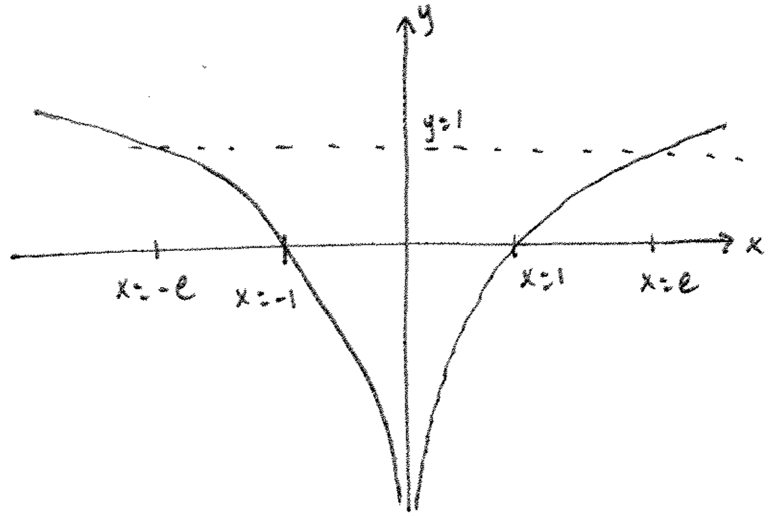
110.107 - Homework #2 Solutions

Section 7.4

15. The integral $\int_{-1}^{1} \ln|x|\,dx$ is improper and converges. Explain why it is improper, and compute its value.

Answer: The graph of $f(x) = \ln|x|$ looks like this:



(pretend my graph is symmetric about the y-axis)

So $\lim_{x \to 0} \ln|x| = \infty$. Since $0 \in [-1, 1]$, the bounds of integration, our integral is improper.

Now, playing around with absolute value signs gives

$$\int_{-1}^{1} \ln|x|\,dx = \lim_{z \to 0^-} \int_{-1}^{z} \ln|x|\,dx + \lim_{z \to 0^+} \int_{z}^{1} \ln|x|\,dx$$

$$= \lim_{z \to 0^-} \int_{-1}^{z} \ln(-x)\,dx + \lim_{z \to 0^+} \int_{z}^{1} \ln x\,dx$$

Let $u = -x$
$du = -dx$
$u(z) = -z$, $u(-1) = 1$

$$= -\lim_{z \to 0^-} \int_{1}^{-z} \ln u\,du + \lim_{z \to 0^+} \int_{z}^{1} \ln x\,dx$$

$$= \lim_{z\to 0^-} \int_{-z}^{1} \ln u \, du + \lim_{z\to 0^+} \int_{z}^{1} \ln x \, dx$$

As $z \to 0^-$, $(-z) \to 0^+$. So we can replace $-z$ with z in the bounds of integration and say $z \to 0^+$.

$$= \lim_{z\to 0^+} \int_{z}^{1} \ln u \, du + \lim_{z\to 0^+} \int_{z}^{1} \ln x \, dx = 2 \lim_{z\to 0^+} \int_{z}^{1} \ln x \, dx$$

(because I know the integral converges, I can add the two).

Now do integration by parts: Let $u = \ln x$, $dv = dx$, $du = \frac{dx}{x}$, $v = x$; we get

$$\int \ln x \, dx = x \ln x - \int x \, \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + C.$$

So our improper integral becomes

$$2 \lim_{z\to 0^+} \int_{z}^{1} \ln x \, dx = 2 \lim_{z\to 0^+} \left[x \ln x - x \right]_{z}^{1}$$

$$= 2 \lim_{z\to 0^+} \left[1 \cdot \ln 1 - 1 - (z \ln z - z) \right]$$

$$= -2 - 2\lim_{z\to 0^+} z \ln z + 2 \underbrace{\lim_{z\to 0^+} z}_{\to 0}$$

We can calculate $\lim_{z\to 0^+} z \ln z$ via L'Hopital's rule:

$$\lim_{z\to 0^+} z \ln z = \lim_{z\to 0^+} \frac{\ln z}{\frac{1}{z}} \overset{\left(\frac{-\infty}{\infty}\right)}{=} \lim_{z\to 0^+} \frac{\frac{d}{dz}[\ln z]}{\frac{d}{dz}\left[\frac{1}{z}\right]} = \lim_{z\to 0^+} \frac{\frac{1}{z}}{-\frac{1}{z^2}}$$

$$= \lim_{z\to 0^+} -z = 0.$$

So the improper integral $\boxed{\int_{-1}^{1} \ln|x| \, dx = -2}$.

Determine whether each integral is convergent. If so, compute its value. ③

22. $\displaystyle\int_0^2 \frac{1}{(x-1)^4}\,dx.$

<u>Answer</u>: As $x \to 1$, the integrand goes to infinity; so our integral is,

$$\int_0^2 \frac{1}{(x-1)^4}\,dx = \lim_{z\to 1^-}\int_0^z \frac{1}{(x-1)^4}\,dx + \lim_{z\to 1^+}\int_z^2 \frac{1}{(x-1)^4}\,dx$$

$$= \lim_{z\to 1^-}\left[-\frac{1}{3}(x-1)^{-3}\right]_0^z + \lim_{z\to 1^+}\left[-\frac{1}{3}(x-1)^{-3}\right]_z^2$$

$$= \lim_{z\to 1^-}\left[-\frac{1}{3}(z-1)^{-3} - \left(-\frac{1}{3}(-1)^{-3}\right)\right]$$

$$+ \lim_{z\to 1^+}\left[-\frac{1}{3}(2-1)^{3} - \left(-\frac{1}{3}(z-1)^{-3}\right)\right]$$

$$= -\frac{1}{3} - \frac{1}{3} + \lim_{z\to 1^+}\frac{1}{3}(z-1)^{-3} - \lim_{z\to 1^-}\frac{1}{3}(z-1)^{-3}$$

$$= -\frac{2}{3} - \frac{1}{3}\underbrace{\lim_{z\to 1^-}\frac{1}{(z-1)^3}}_{\to -\infty} + \frac{1}{3}\underbrace{\lim_{z\to 1^+}\frac{1}{(z-1)^3}}_{\to \infty}$$

Since both limits diverge, the improper integral diverges.

26. $\displaystyle\int_1^e \frac{dx}{x\ln x}.$

<u>Answer</u>: As $x \to 1$, $\ln x \to 0$, which means $\dfrac{1}{x\ln x} \to \infty$.

$$\int_1^e \frac{dx}{x\ln x} = \lim_{z\to 1^+}\int_z^e \frac{dx}{x\ln x} = \lim_{z\to 1^+}\int_{\ln z}^1 \frac{du}{u} = \lim_{z\to 1^+}\Big[\ln|u|\Big]_{\ln z}^1$$

$$= 1 - \lim \ln|\ln z| = 1 - \lim_{z\to 0^+}\ln z = \infty.$$ So the integral diverges.

34. In this problem, we investigate the integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

for $0<p<\infty$.

(a) Compute $\int \frac{1}{x^p}\,dx$ for $0<p<\infty$.

Answer: When $p=1$, we have

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

When $p\neq 1$, we have

$$\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p} + C.$$

(b) Use your result in (a) to compute $\int_c^1 \frac{1}{x^p}\,dx$ for $0<c<1$.

Answer:

When $p=1$,

$$\int_c^1 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_c^1 = \ln 1 - \ln c = \ln\left(\frac{1}{c}\right).$$

When $p\neq 1$,

$$\int_c^1 \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_c^1 = \frac{1}{1-p}\left[1-c^{1-p}\right].$$

(c) Use your result in (b) to show that

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} \quad \text{for } 0<p<1.$$

Answer:

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \frac{1}{1-p}\left[1-c^{1-p}\right]$$

$$= \frac{-1}{1-p}\lim_{c\to 0^+} c^{1-p} + \frac{1}{1-p}.$$

When $0<p<1$, $1-p>0$, so $\lim\limits_{c\to 0^+} c^{1-p} = 0$.

Thus $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$.

(d) Show that

$$\int_0^1 \frac{1}{x^p}\,dx \quad \text{is divergent for } p\ge 1.$$

Answer: When $p=1$,

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{dx}{x} = \lim_{c\to 0^+} \ln\left(\frac{1}{c}\right) = \infty,$$ so the integral diverges.

When $p>1$,

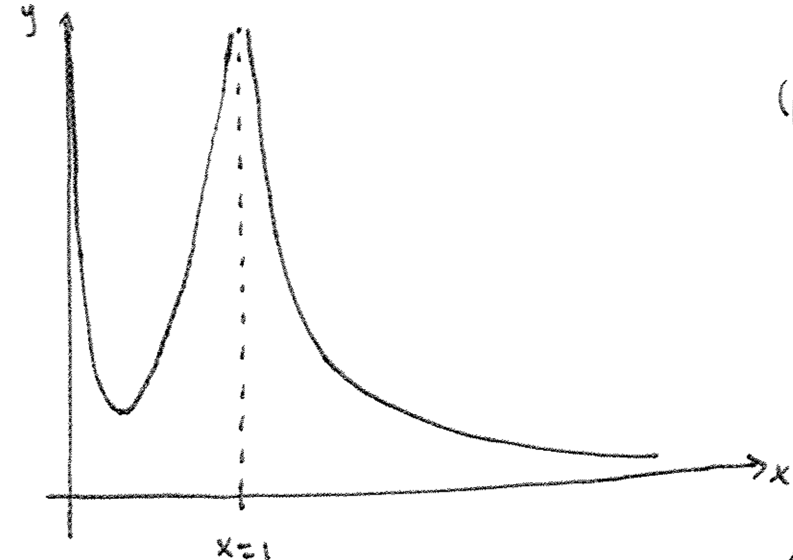
$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{c\to 0^+}\int_c^1 \frac{dx}{x^p} = \lim_{c\to 0^+} \frac{1}{1-p}\left[1-c^{1-p}\right]$$

$$= -\frac{1}{1-p}\lim_{c\to 0^+} c^{1-p} + \frac{1}{1-p}.$$

When $p>1$, $1-p<0$, so $\lim\limits_{c\to 0^+} c^{1-p} = \infty$, which means the integral diverges.

EP3: Determine if the improper integral $\int_0^\infty \frac{1}{x(\ln x)^2}\,dx$ exists or not. If it exists, determine its value.

Answer: The graph of $f(x) = \frac{1}{x(\ln x)^2}$ looks like this:



(pretend the x-axis doesn't curve upward)

If you work out the limits, you'll see that $\lim_{x\to 0^+} f(x) = \lim_{x\to 1} f(x) = \infty$.

So we have to compute four different integrals: one from 0 to c, where $c \in (0,1)$; one from c to 1; one from 1 to d, where $d \in (1,\infty)$; and one from d to ∞. Since we have a natural log in our function, we might as well choose $c = \frac{1}{e}$ and $d = e$. The improper integral becomes

$$\int_0^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{z\to 0^+}\int_z^{1/e} \frac{1}{x(\ln x)^2}\,dx + \lim_{z\to 1^-}\int_{1/e}^{z} \frac{1}{x(\ln x)^2}\,dx$$

$$\lim_{z\to 1^+}\int_z^{e} \frac{1}{x(\ln x)^2}\,dx + \lim_{z\to \infty}\int_e^{z} \frac{1}{x(\ln x)^2}\,dx$$

For the improper integral to exist, all four of these integrals must exist.

Let's do the indefinite integral first:

$$\int \frac{dx}{x(\ln x)^2} \quad \Bigg| \quad \text{Let } u = \ln x, \quad du = \frac{dx}{x} \quad ; \quad \text{the integral becomes}$$

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{\ln x} + C.$$

I'll evaluate all four improper integrals:

$$\lim_{z\to 0^+} \int_z^{1/e} \frac{dx}{x(\ln x)^2} = \lim_{z\to 0^+} \left[-\frac{1}{\ln x}\right]_z^{1/e} = \lim_{z\to 0^+} \left[-\frac{1}{\ln\left(\frac{1}{e}\right)} - \left(-\frac{1}{\ln z}\right)\right]$$

$$= -\frac{1}{(-\ln e)} + \underbrace{\lim_{z\to 0^+} \frac{1}{\ln z}}_{\to 0} = 1.$$

$$\lim_{z\to 1^-} \int_{1/e}^{z} \frac{dx}{x(\ln x)^2} = \lim_{z\to 1^-} \left[-\frac{1}{\ln x}\right]_{1/e}^{z} = \lim_{z\to 1^-} \left[-\frac{1}{\ln z} - \left(-\frac{1}{\ln \frac{1}{e}}\right)\right]$$

$$= -\Bigg(\underbrace{\lim_{z\to 1^-} \frac{1}{\ln z}}_{\to -\infty}\Bigg) - 1 = \infty.$$

Since that integral diverged, the whole integral diverges. So $\int_0^\infty \frac{1}{x(\ln x)^2}\,dx$ <u>does not exist</u>. I'll do the other two integrals anyway:

$$\lim_{z\to 1^+} \int_z^{e} \frac{1}{x(\ln x)^2}\,dx = \lim_{z\to 1^+} \left[-\frac{1}{\ln x}\right]_z^{e} = \lim_{z\to 1^+} \left[-\frac{1}{\ln e} - \left(-\frac{1}{\ln z}\right)\right]$$

$$= -1 + \lim_{z\to 1^+} \frac{1}{\ln z} = \infty.$$ So this integral also diverges.

$$\lim_{z\to\infty} \int_e^{z} \frac{1}{x(\ln x)^2}\,dx = \lim_{z\to\infty} \left[-\frac{1}{\ln x}\right]_e^{z} = \lim_{z\to\infty} \Bigg(\underbrace{-\frac{1}{\ln z}}_{\to 0}\Bigg) - \left(-\frac{1}{\ln e}\right) = 1.$$

Section 8.1

Solve the following pure-time differential equations:

4. $\frac{dy}{dx} = \frac{1}{1+x^2}$, where $y_0 = 1$ when $x_0 = 0$

Answer: Use the separation of variables method; here, all we have to do is integrate both sides with respect to x:

$$\int \frac{dy}{dx}\, dx = \int \frac{dx}{1+x^2}$$

$$y = \tan^{-1}(x) + C.$$

To solve for C, use the initial condition: $y = 1$ when $x = 0$:

$$1 = \tan^{-1}(0) + C \rightarrow C = 1.$$

So the solution is $\boxed{y = \tan^{-1}x + 1}$.

6. $\frac{dx}{dt} = \cos(2\pi(t-3))$, $x(3) = 1$.

Answer: Just integrate both sides w.r.t. t:

$$\int \frac{dx}{dt}\, dt = \int \cos(2\pi(t-3))\, dt \quad \Bigg| \quad \begin{array}{l} \text{Let } u = 2\pi(t-3) \\ du = 2\pi dt, \text{ or } dt = \frac{du}{2\pi} \end{array}$$

$$x = \int \cos u \,\frac{du}{2\pi} = \frac{1}{2\pi}\sin u + C = \frac{1}{2\pi}\sin(2\pi(t-3)) + C.$$

Use the initial condition to solve for C:

$$1 = x(3) = \frac{1}{2\pi}\sin(2\pi(3-3)) + C$$

$1 = 0 + C \rightarrow C = 1$. The solution is $\boxed{x = \frac{1}{2\pi}\sin(2\pi(t-3)) + 1}$.

10. Suppose that the amount of phosphorous in a lake at time t, denoted by $P(t)$, follows the equation

$$\frac{dP}{dt} = 3t + 1 \quad \text{with} \quad P(0) = 0.$$

Find the amount of phosphorous at time $t = 10$.

Answer: First we have to solve the differential equation:

$$\int \frac{dP}{dt}\, dt = \int (3t+1)\, dt$$

$$P(t) = \frac{3}{2}t^2 + t + C$$

The initial condition gives

$$0 = P(0) = \frac{3}{2}(0)^2 + (0) + C \longrightarrow C = 0.$$

So $P(t) = \frac{3}{2}t^2 + t$, and $P(10) = \frac{3}{2}(10)^2 + 10 = \boxed{160}$.

14. Solve the autonomous differential equation

$$\frac{dx}{dt} = 1 - 3x, \quad x(-1) = -2.$$

Answer: Divide by $1-3x$ on both sides:

$$\frac{1}{1-3x}\frac{dx}{dt} = 1$$

$$\int \frac{1}{1-3x}\frac{dx}{dt}\, dt = \int dt$$

The integral on the left is equivalent to $\int \frac{1}{1-3x}\, dx$

$$\int \frac{dx}{1-3x} = t + C_1$$

$$-\frac{1}{3}\ln|1-3x| = t + C_1$$

$$\ln|1-3x| = -3(t+c_1) = -3t + C_2$$

$$e^{\ln|1-3x|} = e^{-3t+C_2}$$

$$|1-3x| = e^{C_2}e^{-3t}$$

$$1-3x = \pm e^{C_2}e^{-3t} = C_3 e^{-3t}, \quad C_3 \in \mathbb{R}$$

$$x = \frac{1}{3} + Ce^{-3t}.$$

Now use $x(-1) = -2$:

$$-2 = x(-1) = \frac{1}{3} + Ce^{-3(-1)} = \frac{1}{3} + Ce^{3}$$

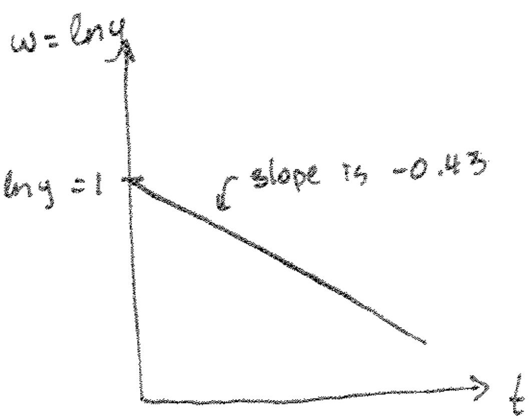
$$Ce^{3} = -2 - \frac{1}{3} = -\frac{7}{3} \rightarrow C = -\frac{7}{3}e^{-3}.$$

So $x(t) = \frac{1}{3} - \frac{7}{3}e^{-3}e^{-3t}$

or

$x(t) = \frac{1}{3} - \frac{7}{3}e^{-3(t+1)}$

18. It helps to draw the graph the book is talking about:



The vertical axis w is equal to $\ln y$ (that's what it means to be a semilog plot).

The equation of the line is

$w = 1 - 0.43t$, and the slope is

$\frac{dw}{dt} = -0.43$. But $\frac{dw}{dt} = \frac{d}{dt}(\ln y) = \frac{1}{y}\frac{dy}{dt}$.

So our differential equation is $\frac{1}{y}\frac{dy}{dt} = -0.43$, or $\frac{dy}{dt} = -0.43y$.

20. (a) We have the differential equation

$$\frac{dW}{dt} = -\lambda W(t), \quad W(0) = W_0$$

Solve it as we have before:

$$\frac{1}{W(t)}\frac{dw}{dt} = -\lambda$$

$$\int \frac{1}{W(t)}\frac{dw}{dt}\,dt = \int (-\lambda)\,dt$$

$$\int \frac{dw}{W(t)} = -\lambda t + C_1$$

$$\ln|W(t)| = -\lambda t + C_1$$

$$e^{\ln|W(t)|} = e^{-\lambda t + C_1}$$

$$|W(t)| = e^{C_1}e^{-\lambda t}$$

$$W(t) = \pm e^{C_1}e^{-\lambda t} = Ce^{-\lambda t}, \quad C \in \mathbb{R}.$$

$$W_0 = W(0) = Ce^0 = C.$$

So $\boxed{W(t) = W_0 e^{-\lambda t}}$.

(b) The half-life τ is the time it takes for the population to decrease by half. More formally,

$$\frac{W_0}{2} = W_0 e^{-\lambda\tau}$$

$$\frac{1}{2} = e^{-\lambda\tau}$$

$$\ln\frac{1}{2} = \ln e^{-\lambda\tau} = -\lambda\tau$$

So $\tau = \dfrac{\ln\frac{1}{2}}{-\lambda} = \dfrac{\ln 2}{\lambda}$.

We have that $W(0) = 123\,g$ and $W(5) = 20\,g$.

So $W(t) = (123g)e^{-\lambda t}$. When $t = 5$ min, we have

$$W(5) = 20g = (123g)e^{-\lambda(5\text{min})}$$

$$\frac{20}{123} = e^{-\lambda(5\text{min})}$$

$$\ln\left(\frac{20}{123}\right) = -\lambda(5\text{ min}) \rightarrow \boxed{\lambda = \frac{1}{5}\ln\left(\frac{123}{20}\right)}.$$

Thus the half-life is

$$\tau = \frac{\ln 2}{\lambda} = \frac{\ln 2}{\frac{1}{5}\ln\left(\frac{123}{20}\right)} = \boxed{\frac{5\ln 2}{\ln\left(\frac{123}{20}\right)}}.$$

21. (a) We have the equation

$$\frac{dN}{dt} = \frac{1}{100}N^2, \quad N(0) = 10.$$

Solve it as before:

$$\frac{1}{N^2}\frac{dN}{dt} = \frac{1}{100}$$

$$\int \frac{1}{N^2}\frac{dN}{dt}dt = \int \frac{dt}{100}$$

$$\int \frac{dN}{N^2} = \frac{1}{100}t + C$$

$$-\frac{1}{N} = \frac{1}{100}t + C$$

$$N = -\frac{1}{\frac{1}{100}t + C}.$$

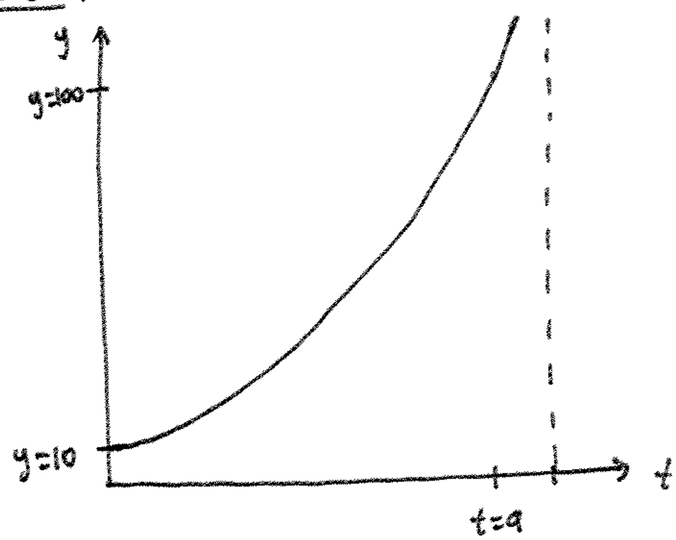
So when $t=0$, $N=10$:

$$10 = -\frac{1}{0+C} \rightarrow C = -\frac{1}{10}. \text{ So}$$

$$\boxed{N(t) = -\frac{1}{\frac{1}{100}t - \frac{1}{10}} = \frac{100}{10-t}.}$$

(b) Graph $N(t)$ as a function of t for $0 \le t < 10$. What happens as $t \to 10$? Explain in words what this means.

Answer: $N(t)$ looks like this:



Since

$$\lim_{t \to 10^-} N(t) = \lim_{t \to 10^-} \frac{100}{10-t} = \infty,$$

the population starts growing without bound as $t \to 10$.

(Obviously this can't happen in a real population.)

26. Use the partial fraction method to solve

$$\frac{dy}{dx} = (y)(1-y), \quad y(0) = 2.$$

Answer:

$$\frac{1}{y(1-y)} \frac{dy}{dx} = 1$$

$$\int \frac{1}{y(1-y)} \frac{dy}{dx}\, dx = \int dx$$

$$\int \frac{dy}{y(1-y)} = x + C_1$$

Now use partial fractions:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y} = \frac{A(1-y) + By}{y(1-y)} = \frac{(B-A)y + A}{y(1-y)}.$$

We get the equations

$$\begin{cases} B - A = 0 \\ A = 1 \end{cases} \rightarrow B = A = 1.$$

So $\int \frac{dy}{y(1-y)} = \int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln|y| - \ln|1-y| + C = \ln\left|\frac{y}{1-y}\right| + C.$

Substituting in,

$$\ln\left|\frac{y}{1-y}\right| = x + C_1$$

$$e^{\ln\left|\frac{y}{1-y}\right|} = e^{x+C_1}$$

$$\frac{y}{1-y} = \pm e^{C_1} e^{x} = Ce^{x}$$

$$y = (1-y)Ce^{x}$$

$$y(1+Ce^{x}) = Ce^{x}$$

$$y = \frac{Ce^{x}}{1+Ce^{x}} \rightarrow 2 = \frac{Ce^{0}}{1+Ce^{0}}$$

$$2 + 2C = C, \quad C = -2$$

So $\boxed{y = \dfrac{-2e^{x}}{1-2e^{x}}.}$

28. Use the partial fraction method to solve

$$\frac{dy}{dx} = (y-1)(y-2), \quad y(0) = 0.$$

<u>Answer</u>:

$$\frac{1}{(y-1)(y-2)}\frac{dy}{dx} = 1$$

$$\int \frac{1}{(y-1)(y-2)}\frac{dy}{dx}\,dx = \int dx$$

$$\int \frac{dy}{(y-1)(y-2)} = x + C_1$$

Now use partial fractions:

$$\frac{1}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2} = \frac{A(y-2)+B(y-1)}{(y-1)(y-2)}$$

$$= \frac{(A+B)y + (-2A-B)}{(y-1)(y-2)}.$$

We get the equations

$$\begin{cases} A+B = 0 \\ -2A-B = 1 \end{cases} \rightarrow \quad A = -B, \text{ so } -2(-B)-B = 1, \text{ or } B = 1.$$

Thus $A = -1$, and

$$\frac{1}{(y-1)(y-2)} = -\frac{1}{y-1} + \frac{1}{y-2}.$$

Now, back to the integral:

$$\int \frac{dy}{(y-1)(y-2)} = -\int \frac{dy}{y-1} + \int \frac{dy}{y-2} = -\ln|y-1| + \ln|y-2| + C$$

$$= \ln\left|\frac{y-2}{y-1}\right|.$$

Substituting that into the differential equation gives

$$\ln\left|\frac{y-2}{y-1}\right| = x + C_1.$$

$$e^{\ln\left|\frac{y-2}{y-1}\right|} = e^{x+C_1}$$

$$\left|\frac{y-2}{y-1}\right| = e^{C_1}e^{x}$$

$$\frac{y-2}{y-1} = Ce^{x}$$

$$y-2 = Ce^{x}(y-1)$$

$$y(1-Ce^{x}) = 2-Ce^{x}$$

$$y = \frac{2-Ce^{x}}{1-Ce^{x}}.$$ The initial condition gives

$$0 = y(0) = \frac{2-Ce^{0}}{1-Ce^{0}} = \frac{2-C}{1-C},$$ so $C = 2$.

The solution is $\boxed{y = \dfrac{2-2e^{x}}{1-2e^{x}}}$.

32. Solve the differential equation

$$\frac{dy}{dx} = (1+y)^2.$$

<u>Answer</u>:

$$\frac{1}{(1+y)^2}\frac{dy}{dx} = 1$$

$$\int \frac{1}{(1+y)^2}\frac{dy}{dx}\,dx = \int dx$$

$$\int \frac{dy}{(1+y)^2} = x + C_1$$

Now, do NOT use a partial fraction decomposition! Just let

$u = 1+y$, $du = dy$; the integral becomes

$$\int \frac{dy}{(1+y)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{1+y} + C.$$

Substituting back in gives

$$-\frac{1}{1+y} = x + C_1.$$

$$1+y = -\frac{1}{x+C_1}$$

$$\boxed{y = -\frac{1}{x+C_1} - 1}.$$